

PROCEEDINGS
OF
THE PHYSICAL SOCIETY
OF LONDON.

JANUARY 1887.

XIX. *The peculiar Sunrise-Shadows of Adam's Peak in Ceylon.*
*By the Hon. RALPH ABERCROMBY, F.R. Met. Soc.**

THERE are certain peculiarities about the shadows of Adam's Peak which have long attracted the attention of travellers : a good deal has been written about them, and several theories have been proposed to explain the observed phenomena. In the course of a meteorological tour round the world, the author stopped in Ceylon for the express purpose of visiting the Peak, and was fortunate enough to see the shadow under circumstances which could leave no doubt as to the true explanation, and which also entirely disproved certain theories which have been propounded on the subject.

The following account is taken from a paper by the Rev. R. Abbay, many years resident in the island, entitled "Remarkable Atmospheric Phenomena in Ceylon," which was read before the Physical Society of London, May 27, 1876, and published in the Philosophical Magazine for July 1876. Writing from descriptions, for he himself had never witnessed the appearance, Mr. Abbay says:—At sunrise apparently an enormous elongated shadow of the mountain is projected to the

* Read November 13, 1886.

westward, not only over the land but over the sea, to a distance of 70 or 80 miles. As the sun rises higher, the shadow rapidly approaches the mountain, and appears at the same time to rise before the spectator in the form of a gigantic pyramid. Distant objects—a hill or a river (or even Colombo itself, at a distance of 45 miles)—may be distinctly seen through it; so that the shadow is not really a shadow on the land, but a veil of darkness suspended between the observer and the low country. All this time it is rapidly rising and approaching, and each instant becoming more distinct, until suddenly it seems to fall back on the spectator, like a ladder that has been reared beyond the vertical; and the next instant the appearance is gone. For this the following explanation is proposed:—The average temperature at night in the low country during the dry season is between 70° and 80° F., whilst that on the summit of the Peak is from 30° to 40° . Consequently the lower strata of air are much less dense than the upper; and an almost horizontal ray of light passing over the summit must of necessity be refracted upwards and suffer total internal reflection as in the case of an ordinary mirage.

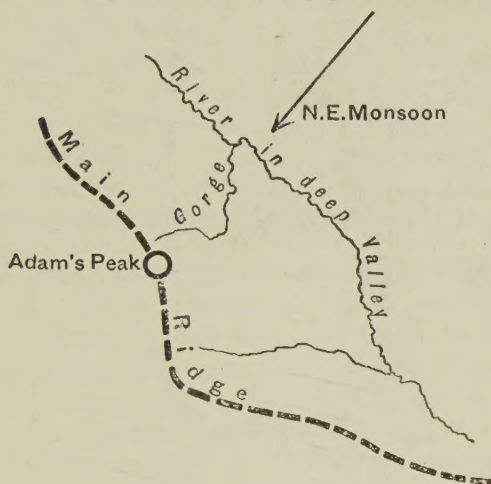
It will be remarked that Mr. Abbay does not allow for the difference of elevation, and the sequel will show that this theory cannot be maintained.

Adam's Peak is a mountain that rises in an abrupt cone, more than 1000 feet above the irregular chain to which it belongs; the summit reaches to 7352 feet above the sea. On the south side the mountain falls suddenly down to Ratnapura, very little above the sea-level; while on the north it slopes irregularly to the high valley of the Maskeliya district. The peak also lies near an elbow in the main chain of mountains, as shown in the diagram of the topography of the Peak (fig. 1), while a gorge runs up from the north-east just to the west of the mountain. When, then, the north-east monsoon blows morning mist up the valley, light wreaths of condensed vapour will pass to the west of the Peak and catch the shadow at sunrise only, if other things are suitable. The importance of this will appear later on.

The only difficulty in getting to Adam's Peak is the want of a rest-house within reasonable distance of the summit. Fortunately the kindness and hospitality of T. N. Christie,

Esq., of St. Andrew's Plantation, Maskeliya, enabled the author, in company with Mr. G. Christie and Professor Bower, of the University of Glasgow, to make the ascent with great comfort and with a few necessary instruments. Our party reached the summit on the night of the 21st February, 1886,

Fig. 1.—Diagram of the Topography of Adam's Peak.



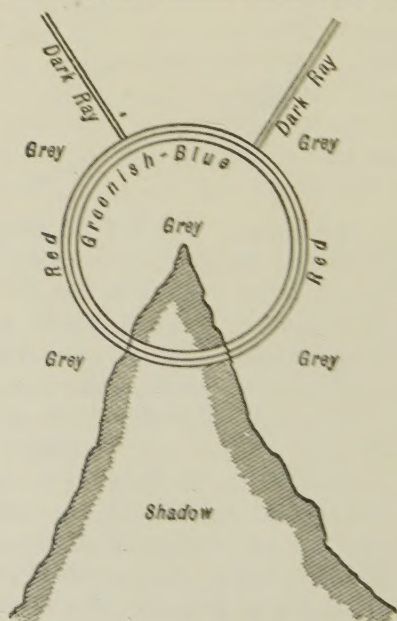
amid rain, mist, and wind. Towards morning the latter subsided, but at 5.30 A.M. the sky was covered with a confused mass of nearly every variety of cloud. Below and around us cumulus and mist; at a higher level, pure stratus; above that, wild cirro-stratus and fleecy cirro-cumulus.

Soon the foreglow began to brighten the under surface of the stratus-cloud with orange; lightning flickered to the right of the rising sun over a dense mass of cloud; opposite, a light pink-purple illumined an irregular layer of condensed vapour; while above, a pale moon with a large ill-defined *corona* round her struggled to break through a softish mass of fleecy cloud. Below lay the island of Ceylon, the hills and valleys presenting the appearance of a raised relief-map; patches of white mist filled the hollows; true cloud drove at intervals across the country, and sometimes masses of mist coming up from the valley enveloped us with condensed vapour.

At 6 A.M. the thermometer marked 52° F.; we had been told that the phenomenon of the shadow depended on the

temperature at the summit falling to 30° or 40° F.; and when, shortly after, the sun rose behind a cloud we had almost lost all hope of seeing anything; but suddenly at 6.30 A.M. the sun peeped through a chink in the clouds, and we saw the pointed shadow of the Peak lying on the misty land. Driving condensed vapour was floating about, and a fragment of rainbow-tinted mist appeared near the top of the shadow. Soon this fragment grew into a complete prismatic circle of about 8° diameter by estimation, with the red outside, formed round the summit of the Peak as a centre. The author instantly saw that with this bow there ought to be spectral figures, so he waved his arms about and immediately found shadowy arms moving in the centre of the rainbow. Two dark rays shot upwards and outwards on either side

Fig. 2.—Diagram of rainbow round the shadow.



of the centre, as shown in the diagram fig. 2, and appeared to be nearly in a prolongation of the lines of the slope of the Peak below. The centre of the bow appeared to be just below the point of the shadow, not on it; because we were

standing on a platform below a pointed shrine, and the subjective bow centred from our own eyes. If we did not stand fairly out in the sun, only a portion of the bow could be seen. Three times, within a quarter of an hour, this appearance was repeated as mist drove up in proper quantities, and fitful glimpses of the sun gave sufficient light to throw a shadow and form a bow. In every case the shadow and bow were seen in front of land and never against the sky. The last time, when the sun was pretty high, we saw the characteristic peculiarity of the shadow. As a thin wreath of condensed vapour came up from the valley at a proper height, a bow formed round the shadow, while both seemed to stand up in front of us, and then the shadow fell down on to the land, and the bow vanished as the mist passed on.

Here, then, was an unequivocal explanation of the whole phenomenon. The apparent upstanding of the shadow was simply the effect of passing mist which caught the darkness of the Peak at a higher level than the earth, for when the condensed vapour moved on, the characteristic bow disappeared and the shadow fell to its natural plane on the ground. When the mist was low, as on the two first occasions, the shadow fell on the top as it were, and there was no appearance of lifting, only the formation of a bow.

The well-known theory of the bow is that light diffracted in its passage between small water-globules forms a series of bows according to the size of the globules, their closeness, and the intensity of the illumination. Had the mist been so fine and thin as merely to catch and raise the shadow, but not to form a bow, there might have been some doubt as to the origin of the appearance. Our fortune was in the unsettled weather, which made the mist so coarse and close that the unequivocal bow left no doubt as the true nature of the cause.

About an hour later the sun again shone out, but much higher and stronger than before, and then we saw a brighter and sharper shadow of the Peak, this time encircled by a double bow. Our own spectral arms were again visible, but the shadow was now so much nearer the base of the Peak, and we had to look so much down on it, that there was no illusion of standing up, and there were no dark diverging

rays. The inner bow was the one we had seen before; the outer and fainter one was due to stronger light.

The bows were all so feeble and the time so short, that the author did not succeed in obtaining any sextant measurements of the diameters of the bows; but his thermometric observations conclusively disprove any idea of mirage. At 6 A.M. the thermometer on the Peak marked 52° F., while at Colombo the temperature stood at $74^{\circ}85$. The difference of $22^{\circ}85$ is just about the normal difference in temperature due to a height of 7352 feet.

The Colombo figures were procured through the courtesy of the Surveyor-General for Ceylon. They are got as follows:—Colombo observations only give the *minimum* that morning as $73^{\circ}6$ F., and the 7 A.M. reading as $75^{\circ}5$. The mean curve of diurnal temperature for the month of February, as determined by the Office, gives a difference of $0^{\circ}65$ between the 6 A.M. and 7 A.M. observations; and by subtracting that correction from $75^{\circ}5$ we get $74^{\circ}85$ as the 6 A.M. reading.

The questions have been frequently asked—Why this lifted shadow should be peculiar to Adam's Peak?; why a similar appearance is not observed from any other mountain-top?; and why the shadow is rarely seen at sunset? There are not many mountains which are habitually visited that are either over 7000 feet, or that rise in an isolated, well-defined pyramid. Still fewer can there be where a steady wind, for months together, blows up a valley so as to project the rising morning mist at a suitable height and distance on the western side to catch the shadow of the peak at sunrise. The shadow is not seen during the south-west monsoon, for then the mountain is covered with cloud and deserted. Nowhere either do we find at sunset those light mists lying near the ground which are so characteristic of sunrise, and whose presence is necessary to lift the shadow.

The combination of a high isolated pyramid, a prevailing wind, and a valley to direct suitable mist at a proper height on the western side of the mountain, is probably only rarely met with; and at present nothing yet has been described that exactly resembles this sunrise shadow of Adam's Peak in the green island of Ceylon.

But there is another totally different shadow which is sometimes seen from Adam's Peak, just before and at the moment of sunrise, that has been mixed up in some accounts with the shadow we have just described. The shadow of the base of the Peak stretches along the land to the horizon, and then the shadow of the summit appears to rise up and stand against the distant sky. The first part seems to be the natural shadow lying on the ground ; and the sky part to be simply the ordinary earth shadow of twilight projected so clearly against the sky as to show mountainous irregularities of the earth's surface. As the sun rises, the shadow of the summit against the sky gradually sinks to the horizon, and then the ordinary shadow grows steadily shorter as the sun gets higher in the usual manner. This can only be seen at sunrise from Adam's Peak, because the ground to the east is too high and mountainous to allow the shadow of the summit to fall on the sky before the sun is too far down.

The author found a similar effect, only at sunset, on Pike's Peak, Colorado, 14,147 feet above the sea, and nearly double the height of Adam's Peak. There, towards sunset, the shadow of the mountain creeps along the level prairie to the horizon, and there begins to rise up in the sky till the sun has just gone down, and the anticrepuscular shadow rises too high to catch the outline of the Peak. The author only witnessed a portion of this sequence, for just about the time that the shadow stretched to the horizon, clouds obscured the sun, and the rise of the shadow could not be observed ; but from all the descriptions he heard, there can be no doubt that the character of the shadow is identical with that of Adam's Peak, only that, as the order of sequence is reversed, it is more easy to follow the origin of the shadows.

Since the above was written, the author's attention has been called to the sketch of the shadow exhibited by the well-known traveller Miss C. F. Gordon Cumming, in the Colonial Exhibition. This picture represents the shadow lying down, but not raised, on an irregular surface of white mist and mountain tops. The most interesting thing is a prismatic fringe of colour along the straight outside edges of the shadow ; but there is no trace of a bow round its point.

When we consider how much the appearance of the shadow

depends on the height, size, and aggregation of the mist, we need not be surprised at the numerous phases of reflection and refraction that have been described by travellers; but the general principles which have been laid down in this paper appear to govern all.

XXI. *On the Critical Mean Curvature of Liquid Surfaces of Revolution.* By A. W. RÜCKER, M.A., F.R.S.*

LET a weightless mass of liquid, or a liquid film, be attached to two equal circular rings, the planes of which are perpendicular to the line joining their centres. It will form a surface of revolution; and if it is in stable equilibrium, the longest or the shortest diameter will be half way between the rings. It is convenient to call this the *principal diameter*. At all points on the surface the sum of the reciprocals of the two principal radii of curvature is constant. Half this quantity may be called the *mean curvature*. Maxwell has, in his article on Capillary Action (*Enc. Brit.*, 9th edition), given a simple proof of the fact that if the film is a cylinder, a slight bulge will cause an increase or decrease in the mean curvature according as the distance between the rings is less or greater than half the circumference of either. If the distance between the rings is exactly half the circumference, an infinitely small change in the volume will modify the form of the surface, but will not alter the mean curvature. Thus the mean curvature of a cylinder, the length of which has this particular ratio ($\pi/2$) to its diameter, is evidently a maximum or minimum with respect to that of other surfaces of constant mean curvature, which pass through the same rings at the same distance apart, and which differ but little from the cylindrical form. Hence the cylinder may be said to have a *critical mean curvature* when the distance between the rings is half their circumference. If the distance between the rings is altered, a similar property will be possessed by some other surface. It is proposed in the present paper to determine the general relation between the magnitude and distance of the rings and the form of the surfaces of critical curvature.

* Read November 27, 1886.

The expression for the change in the mean curvature of a film or liquid mass, under the conditions above laid down, has been investigated in a paper "On the Relation between the Thickness and the Surface-tension of Liquid Films," lately communicated by Prof. Reinold and myself to the Royal Society. It was, however, applied only to the cases which were practically realized in the experiments therein described. It will be convenient, before discussing it more fully, to indicate the manner in which the equation is obtained.

Beer has shown that if the axis of x be the axis of revolution, the equation to a liquid surface of revolution is given by the expressions

$$x = \alpha E + \beta F, \quad y^2 = \alpha^2 \cos^2 \phi + \beta^2 \sin^2 \phi; \quad \dots \quad (1)$$

where F and E are elliptic integrals of the first and second kinds respectively, of which the amplitude is ϕ , and the modulus $\kappa = \sqrt{\alpha^2 - \beta^2}/\alpha$.

As usual

$$\Delta = \sqrt{1 - \kappa^2 \sin^2 \phi}, \quad \dots \quad (2)$$

whence $y = \alpha \Delta$; and if $\kappa = \sin \theta$, $\beta = \alpha \cos \theta$.

Since $\alpha > \beta$, α and β are the maximum and minimum values of y respectively: and the above equations implicitly assume that the origin lies on a maximum ordinate; for when $\phi = 0$, $x = 0$ and $y = \alpha$. If we wish to transform to a minimum ordinate, ϕ is $> \pi/2$, and

$$x = \alpha(E - E_1) + \beta(F - F_1), \quad \dots \quad (1a)$$

where E_1 and F_1 are the complete integrals.

It may be well, for the sake of clearness, to state that the surface is an unduloid or nodoid according as β is positive or negative, *i. e.* according as $\pi/2 > \theta > -\pi/2$ or $3\pi/2 > \theta > \pi/2$.

If θ be supposed to vary continuously, and if one at least of the quantities α and β is finite, the form of the surface may be made to pass through a continuous cycle of changes.

Thus, between $\theta = 0$ and $\theta = \pi/2$ the surface is an unduloid, the limits being the cylinder when $\theta = 0$, and the sphere when $\theta = \pi/2$. As θ passes through the next quadrant the surface is a nodoid, the limits being the sphere, and a circle the plane of which is perpendicular to the axis of revolution, which is, as Plateau points out, a purely mathematical limit. In the third quadrant the surface is again a nodoid, the limits of

which are the circle and catenoid. Finally, when θ lies between $3\pi/2$ and 2π the surface is an unduloid, the limits of which are the catenoid and the cylinder.

If now $2X$ and $2Y$ are the distance between and diameter of the rings respectively, and if ϕ_1 is the value of ϕ when $y=Y$, we have

$$X = \alpha E + \beta F, \quad Y^2 = \alpha^2 \cos^2 \phi_1 + \beta^2 \sin^2 \phi_1.$$

Hence if α, β , and ϕ_1 vary, but so that X and Y are unaltered, we have, by differentiation,

$$\begin{aligned} & \left\{ E - \frac{\beta^2}{\alpha^2} \int_0^{\phi_1} \frac{\sin^2 \phi}{\Delta} d\phi + \frac{\beta^3}{\alpha^3} \int_0^{\phi_1} \frac{\sin^2 \phi}{\Delta^3} d\phi \right\} \delta\alpha \\ & + \left\{ \frac{\beta}{\alpha} \int_0^{\phi_1} \frac{\sin^2 \phi}{\Delta} d\phi + F - \frac{\beta^2}{\alpha^2} \int_0^{\phi_1} \frac{\sin^2 \phi}{\Delta^3} d\phi \right\} \delta\beta \\ & + \left\{ \alpha \Delta_1 + \frac{\beta}{\Delta_1} \right\} \delta\phi_1 = 0, \quad \dots \dots \dots (3) \end{aligned}$$

and

$$2\alpha \cos^2 \phi_1 \delta\alpha + 2\beta \sin^2 \phi_1 \delta\beta - (\alpha^2 - \beta^2) \sin 2\phi_1 \delta\phi_1 = 0. \quad (4)$$

But

$$\begin{aligned} \int \frac{\sin^2 \phi}{\Delta} d\phi &= \frac{F-E}{\kappa^2}, \quad \int \frac{\sin^2 \phi}{\Delta^3} d\phi = \frac{1}{\kappa^2} \left\{ \int \frac{d\phi}{\Delta^3} - F \right\}, \\ \int \frac{d\phi}{\Delta^3} &= \frac{\alpha^2}{\beta^2} \left\{ E - \frac{\kappa^2 \sin \phi_1 \cos \phi_1}{\Delta_1} \right\}. \quad \dots \dots (5) \end{aligned}$$

Substituting these values in (3) and (4) and eliminating $\delta\phi_1$ between them we get

$$\left. \begin{aligned} (\alpha^2 E - \beta^2 F + \alpha^2 \Delta_1 \cot \phi_1) \delta\alpha + \alpha^2 (F - E + \Delta_1 \tan \phi_1) \delta\beta &= 0, \\ \text{or} \quad A \delta\alpha + B \delta\beta &= 0. \end{aligned} \right\} \quad (6)$$

Now the mean curvature of a surface of revolution of minimum area has been proved by Lindelöf to be the same as that of a circle of radius $(\alpha + \beta)$. Hence

$$\begin{aligned} \frac{1}{2} \delta \left(\frac{1}{R} + \frac{1}{R'} \right) &= - \frac{\delta\alpha + \delta\beta}{(\alpha + \beta)^2} \\ &= \frac{A-B}{B} \cdot \frac{\delta\alpha}{(\alpha + \beta)^2} = - \frac{A-B}{A} \cdot \frac{\delta\beta}{(\alpha + \beta)^2}. \end{aligned}$$

Hence the mean curvature has in general a critical value when $A-B=0$.

First confining ourselves to the case in which the principal ordinate is a maximum, and ϕ_1 and θ are less than

$\pi/2$, it is evident that, since F is always $> E$, B is always positive.

Also, by (5),

$$\frac{1}{\kappa} \frac{dF}{d\kappa} = \int \frac{\sin^2 \phi}{\Delta^3} d\phi = \frac{1}{\beta^2 \kappa^2} \left\{ \alpha^2 E - \beta^2 F - \alpha^2 \kappa^2 \frac{\sin \phi_1 \cos \phi_1}{\Delta_1} \right\};$$

whence, since $dF/d\kappa$ is positive and $\sin \phi_1 \cos \phi_1$ is positive, $\alpha^2 E - \beta^2 F$ is positive, and therefore A is positive also. Further, B can only vanish if $\phi_1 = 0$; and none of the terms in A or B become infinite unless $\phi_1 = 0$ or $\pi/2$, cases which it will be seen hereafter it is unnecessary to consider.

Thus,

$$(A - B)/\alpha^2 = 2E - F(1 + \cos^2 \theta) + 2\Delta_1 \cot 2\phi_1 = 0 \quad (7)$$

is a relation which must be satisfied by ϕ_1 and θ when the mean curvature has a critical value for changes in the form of the surface which take place, subject to the conditions that the radii of the rings and the distance between them are constant.

Corresponding values of ϕ_1 and θ must be found by trial; but it will now be shown that if such a pair of values is known, when $\pi/2 > \theta > 0$, the values of ϕ_1 which are proper to $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$ can be readily deduced without further trials.

In the first place it is evident that, since the squares of the sine and cosine of θ alone enter into (7), the curve obtained by plotting the values of θ as abscissæ and those of ϕ_1 as ordinates is symmetrical with respect to the ordinate $\theta = \pi/2$, and that the same value of ϕ_1 corresponds both to θ and $\pi - \theta$.

If, then, we conceive a film attached to two rings, the volume and length of which vary continuously in such a way that (7) is always satisfied, as the cylinder changes to the sphere and thus to a nodoid forms which correspond to the same value of $\sin^2 \theta$ will have the same value of ϕ_1 also, and the lengths will be given by the expression

$$X = a(E + \cos \theta F); \quad \dots \dots (8)$$

where θ is $< \pi/2$ for the unduloid and $> \pi/2$, but $< \pi$ for the nodoid.

All these forms will have a maximum diameter half way between the rings. If we now proceed to discuss cases where the principal ordinate is a minimum, we must all

through the previous investigation consider the lower limit of the integrals to be $\pi/2$ instead of 0, and ϕ_1 to be $>\pi/2$. With this convention no change is produced in any of the equations; as in equations (5) the quantities which are brought outside the sign of integration vanish, both when $\phi_1=0$ and when $\phi_1=\pi/2$.

Thus, writing as usual E_1 and F_1 for the complete integrals, and taking ϕ' instead of ϕ_1 as the upper limit of E and F , where ϕ' is $>\pi/2$, we have

$$2(E-E_1)-(F-F_1)(1+\cos^2\theta)+2\Delta(\phi')\cot 2\phi'=0. \quad (9)$$

Let ψ be an angle such that

$$F(\phi')-F_1=F(\psi);$$

then, by the addition theorem,

$$E(\phi')-E_1=E(\psi)-\sin^2\theta\sin\phi'\sin\psi.$$

Also

$$\tan\phi'\tan\psi=-\sec\theta,$$

$$\sin\phi'=\cos\psi/\Delta(\psi),$$

$$\Delta(\phi')\Delta(\psi)=\cos\theta,$$

and

$$\cot 2\phi'=\frac{1-\cos^2\theta\tan^2\psi}{2\cos\theta\tan\psi}.$$

Hence, substituting in (9),

$$\begin{aligned} 2E(\psi)-F(\psi)(1+\cos^2\theta)-2\sin^2\theta\frac{\cos\psi\sin\psi}{\Delta(\psi)} \\ +2\frac{\cos\theta}{\Delta(\psi)}\times\frac{1-\cos^2\theta\tan^2\psi}{2\cos\theta\tan\psi}=0; \end{aligned}$$

or

$$2E(\psi)-F(\psi)(1+\cos^2\theta)+2\Delta(\psi)\cot 2\psi=0, \quad (10)$$

which is the same as (7).

We thus conclude that, for every angle ψ or ϕ_1 which satisfies (7), there is a corresponding angle ϕ' which satisfies (9) for the same value of κ^2 , and that these angles are connected by the relation

$$\tan\phi'\tan\phi_1=-\sec\theta. \quad (11)$$

If, then, we determine from (7) the values of ϕ_1 which correspond to certain values of θ between 0 and $\pi/2$, we can by (11) find the values of ϕ' corresponding to values between $3\pi/2$ and 2π ; whence, since (7) and (9) depend on the

squares of the sine and cosine of θ , the values of ϕ and ϕ' between $\pi/2$ and $3\pi/2$ are also known.

Before making any numerical calculations it is convenient to discuss (7) more fully.

Differentiating, we get

$$\sin 2\theta \left\{ -\int_0^{\phi_1} \frac{\sin^2 \phi}{\Delta} d\phi - \frac{1 + \cos^2 \theta}{2} \int_0^{\phi_1} \frac{\sin^2 \phi}{\Delta^3} d\phi + F - \frac{\sin^2 \phi_1 \cot 2\phi_1}{\Delta_1} \right\} \delta\theta + \left\{ 2\Delta_1 - \frac{1 + \cos^2 \theta}{\Delta_1} - \frac{\sin^2 \theta \cos 2\phi_1}{\Delta_1} - 4\Delta_1 \operatorname{cosec}^2 2\phi_1 \right\} \delta\phi_1 = 0;$$

which, if we use equations (5) and simplify, becomes

$$\tan \theta \{ E - F \cos^2 \theta - \Delta_1 \tan \phi_1 \} \delta\theta + 4\Delta_1 \operatorname{cosec}^2 2\phi_1 \delta\phi_1 = 0. \quad (12)$$

Hence $\delta\phi_1 = 0$ if $\theta = n\pi$, and if $\phi_1 = n\pi/2$. Also, considering the case in which

$$E - F \cos^2 \theta - \Delta_1 \tan \phi_1 = 0, \quad \quad (13)$$

we notice that, if we subtract (13) from (7), we get

$$E - F + \Delta_1 \cot \phi_1 = 0; \quad \quad (14)$$

and these equations are satisfied if $\theta = \pi/2$ and $\phi_1 = 56^\circ 28'$.

For, if $\theta = \pi/2$, (13) is true identically, and (14) reduces to

$$\log_e \tan \left(\frac{\pi}{4} + \frac{\phi_1}{2} \right) = \operatorname{cosec} \phi_1,$$

which holds good when $\phi_1 = 56^\circ 28'$.

Hence, when $\theta = \pi/2$, $\delta\phi_1/\delta\theta$ is of the form $\infty \times 0$, which is readily shown to be equal to zero.

To find the corresponding value of $\delta\phi'/\delta\theta$, we have from (11),

$$\sec^2 \phi_1 \tan \phi' \frac{d\phi_1}{d\theta} + \sec^2 \phi' \tan \phi_1 \frac{d\phi'}{d\theta} = -\sin \theta \sec^2 \theta;$$

and by substituting from (11) for $\tan \phi'$ and $\sec^2 \phi'$, this becomes

$$-\cos \theta \frac{d\phi_1}{d\theta} + \Delta_1^2 \frac{d\phi'}{d\theta} = -\frac{\sin \theta \sin 2\phi_1}{2}.$$

Putting $\theta = \pi/2$,

$$\frac{d\phi'}{d\theta} = -\tan \phi_1 = -\tan 56^\circ 28'.$$

It must be remembered that θ corresponds to ϕ_1 and that if θ' corresponds to ϕ' , $\theta' = 2\pi - \theta$, so that $d\phi'/d\theta' = -d\phi'/d\theta$.

The question as to whether the critical value is a maximum or minimum has not yet been discussed. Since $A - B = 0$,

this depends on the sign of $\frac{d(A-B)}{d\alpha}$; or, if we write $a=A/\alpha^2$, $b=B/\alpha^2$, upon that of $\frac{d}{d\alpha}(a-b)$, where a and b are explicitly functions of θ and ϕ_1 only. Now, putting $a-b=\chi$,

$$\frac{d}{d\alpha}(a-b) = \frac{d\chi}{d\alpha} = \left(\frac{d\chi}{d\theta}\right) \frac{d\theta}{d\alpha} + \left(\frac{d\chi}{d\phi_1}\right) \frac{d\phi_1}{d\alpha},$$

where $\left(\frac{d\chi}{d\theta}\right)$ and $\left(\frac{d\chi}{d\phi_1}\right)$ are the coefficients of $\delta\theta_1$ and $\delta\phi_1$, in (12), with the signs changed.

But since $A=B$, $d\beta/d\alpha = -1$, from (6). Hence, since $\cos\theta = \beta/\alpha$,

$$\sin\theta \frac{d\theta}{d\alpha} = \frac{\alpha + \beta}{\alpha^2}.$$

In like manner, from (4),

$$\frac{d\phi_1}{d\alpha} = \frac{2(\alpha \cos^2 \phi_1 - \beta \sin^2 \phi_1)}{(\alpha^2 - \beta^2) \sin 2\phi_1},$$

so that

$$\frac{d\chi}{d\alpha} = \left(\frac{d\chi}{d\theta}\right) \frac{\alpha + \beta}{\alpha^2 \sin\theta} + 2 \left(\frac{d\chi}{d\phi_1}\right) \frac{\alpha \cos^2 \phi_1 - \beta \sin^2 \phi_1}{(\alpha^2 - \beta^2) \sin 2\phi_1}.$$

Now as we pass from one surface which satisfies the condition $A-B=0$ to another, the value of X changes; and it can easily be shown that if $dX/d\theta$ be calculated subject to this condition, it is of the same sign as $d\chi/d\alpha$. Hence if X increases with θ , $d\chi/d\alpha$ is positive and the critical value is a minimum; if X diminishes as θ increases, it is a maximum. If X is a maximum or minimum the curvature has a stationary value, but it is not itself a maximum or minimum.

I have calculated by trial the values of ϕ_1 which satisfy (7) for a few angles between 0° and 90° . They are given, together with the corresponding values of E , F , and Δ_1 , in Table I.

TABLE I.

θ .	ϕ_1 .	E .	F .	Δ_1 .
0	45°00	0·785	0·785	1·000
10	45·23	0·787	0·792	0·992
30	46·94	0·799	0·841	0·931
45	49·28	0·812	0·913	0·844
60	52·25	0·824	1·018	0·729
80	55·84	0·833	1·169	0·580
90	56·47	0·834	1·200	0·552

The values in the last four columns are repeated in the reverse order as θ increases from 90° to 180° .

In the next Table are given the values of $\phi'' = \pi - \phi'$, and of $E_1 - E(\phi'')$ and $F_1 - F(\phi'')$. In representing the results graphically it is best to take $\phi' - \pi/2$ or ϕ_2 , as corresponding to ϕ_1 , and therefore these values are also given.

TABLE II.

θ .	ϕ'' .	ϕ_2 .	$E_1 - E(\phi'')$.	$F_1 - F(\phi'')$.
360	45.00	45.00	0.785	0.785
350	45.20	44.80	0.772	0.792
330	47.20	42.80	0.664	0.841
315	50.60	39.40	0.519	0.913
300	57.15	32.85	0.326	1.018
280	75.65	14.35	0.055	1.169
270	90.00	0.00	0.000	1.200

The values of $\Delta(\phi')$ are omitted because they are readily obtained by the formula $\Delta(\phi')\Delta(\phi_1) = \cos \theta$.

The curve obtained by means of these Tables, which shows the relation between ϕ_1 or ϕ_2 and θ , is given in fig. I.

Rectangular coordinates are perhaps the most convenient; but if θ and ϕ be regarded as angle and radius vector, the curve assumes the symmetrical form shown in fig. II.

This result completes the solution of the problem; but the nature of the conclusions at which we have arrived is more evident if we proceed to deduce the ratios of the lengths and principal diameters of the films to the radii of the rings.

This is done by means of the following relations, where symbols with unity subscript refer to bulging films, and those with 2 subscript to films the principal ordinate of which is a minimum.

$$\alpha_1/Y = 1/\Delta_1,$$

$$X_1/\alpha_1 = E + F \cos \theta;$$

whence X_1/Y is found.

$$\beta_2/Y = \alpha_2 \cos \theta/Y = \cos \theta/\Delta_2 = \Delta_1,$$

$$X_2/\beta_2 = (E' - E_1) \sec \theta + (F' - F_1);$$

whence X_2/Y is obtained.

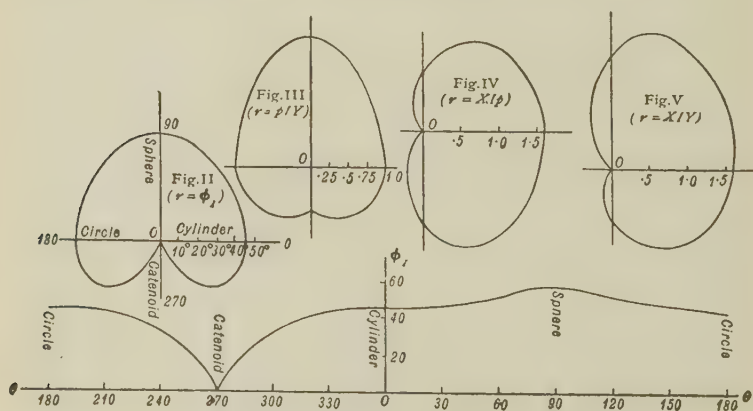
It is evident from these equations that $\alpha_1\beta_2 = Y^2$; i. e. the

radius of the rings is a mean proportional between the principal ordinates of two surfaces in which the modulus of the elliptic integrals is the same, and the principal ordinates of which are a maximum and a minimum respectively.

TABLE III.

θ	$\alpha_1 Y$	$X \alpha_1$	$X Y$	θ	β_2/Y	X/β_2	X/Y
0	1.000	1.571	1.571	180	1.000	0.000	0.000
10	1.008	1.567	1.579	190	0.992	0.008	0.008
30	1.074	1.527	1.640	210	0.931	0.073	0.068
45	1.184	1.458	1.726	225	0.844	0.179	0.151
60	1.372	1.333	1.829	240	0.729	0.366	0.267
80	1.725	1.036	1.787	260	0.580	0.852	0.494
90	1.810	0.834	1.509	270	0.552	1.200	0.663
100	1.725	0.630	1.086	280	0.580	1.486	0.862
120	1.372	0.316	0.433	300	0.729	1.671	1.218
135	1.184	0.166	0.196	315	0.844	1.647	1.391
150	1.074	0.071	0.076	330	0.931	1.607	1.495
170	1.008	0.008	0.008	350	0.992	1.576	1.563
180	1.000	0.000	0.000	360	1.000	1.571	1.571

The "march" of the functions is shown by means of the curves in figs. III., IV., and V. Thus, if p be the length of the principal ordinate (whether it be a maximum or a minimum), fig. III. shows the relation between p/Y and θ , fig. IV. that between X/p and θ , and fig. V. that between X/Y and θ .

Fig. I. ($x=\theta$, $y=\phi_1$).

By plotting the values of $X Y$ we find that the maximum occurs when $\theta = 70^\circ$. The corresponding value of ϕ_1 is $54^\circ 15'$,

and this gives

$$\alpha/Y = 1.545, \quad X/\alpha = 1.2044, \quad X/Y = 1.860.$$

If, then, we suppose the rings to approach to or recede from each other, and the volume and diameter of the film to be at the same time altered so that it always satisfies the conditions of critical mean curvature, it will undergo the following changes of form.

Starting with the rings in contact, and supposing that as they separate the film has a slight bulge, it will first be a nodoid, and the length and principal diameter will increase together. When the length is a little more than one and a half (1.509) diameters of the rings the film is spherical, and the principal diameter is then a maximum ($\alpha/Y = 1.810$). As the diameter begins to decrease the film becomes an unduloid, but the length increases until it is $1.860 \times$ diameter of rings. Thereafter length and diameter decrease together until, when the latter is a third proportional to the diameters of the sphere and of the rings, it reaches its minimum value ($\beta/Y = 0.552$). The film is then a catenoid. As the length diminishes it becomes a nodoid, exerting a negative or outward pressure, and this continues until the cycle is completed by the rings meeting again.

The whole of the above investigation has taken place subject to the condition that $\phi_1 < \pi/2$, and without reference to the stability of the films, which is, however, secured by the condition as to ϕ_1 except in the neighbourhood of $\theta = 180^\circ$.

The curves, when drawn on a larger scale, lend themselves to the solution of a number of problems with an accuracy quite sufficient for practical purposes.

Thus, if we wish to determine the conditions of the film which has a critical curvature when the principal diameter or the length is a given multiple of the diameter of the rings, we have only to draw a circle with the origin in figs. III. or V. as the centre, and with the radius equal to the given ratio. The points of intersection give the value of θ ; ϕ_1 is found from fig. II.; and thus the other quantities can be determined either by calculation or by means of the other figures.

It is evident, since the maximum radius of the curve in fig. V. is such that $X/Y = 1.860$, that the curvature cannot have a critical value for films such that the ratio of the length

to the diameter of the rings exceeds this number, while for all less ratios there must be two critical points, a maximum and a minimum respectively.

If, then, we suppose a film attached to two rings to be initially a nodoid with a diameter exceeding that of the sphere, and to contract gradually, its behaviour, as regards change of curvature, within the limits of the problem, would be as follows.

If the length were $> 1.860 \times$ diameter of rings, the film, after becoming a sphere, would always be an unduloid until it reached the limit at which the conditions no longer apply. The mean curvature would increase as the principal diameter diminished.

If $1.860 > X/Y > 0.663$, the film remains an unduloid throughout all stages after it has become a sphere; but the mean curvature first increases, then diminishes, and finally increases again. The cylinder is the form of minimum mean curvature if $X/Y = 1.571$. The sphere is the form of maximum curvature if $X/Y = 1.509$.

If $X/Y = 0.663$ the last series of statements holds good, with the addition that the minimum mean curvature is zero. Hence the surface passes through the form of the limiting catenoid, which is such that no catenoid can be formed between the rings if the distance between them is increased. If the distance between the rings is diminished, two catenoids pass through them.

If $X/Y < 0.663$, the maximum mean curvature which is attained while the film is still a nodoid diminishes as the figure passes through the forms of the sphere, cylinder, and catenoid, and then becomes negative, *i. e.* the pressure exerted by the film is directed outwards. The minimum is reached when the form of the film lies between the two catenoids which can be drawn through the rings.

The calculations enable us also to solve another problem. If the interiors of two similar films be connected which are formed between equal and equidistant rings, and which are stable when separated from each other, the system will only be in stable equilibrium if a contraction in the principal ordinate, producing a decrease in volume, is attended by a decrease in the curvature.

Hence no pair of similar films so arranged can be in stable equilibrium if the length is $> 1.860 \times$ diameter of rings.

Two cylinders cannot be in stable equilibrium if the length is $> \frac{\pi}{2} \times$ diameter, nor two spheres if the length is $> 1.509 \times$ diameter of rings, *i. e.* $> 0.834 \times$ diameter of sphere.

*XXI. Influence of Change of Condition from the Liquid to the Solid State on Vapour-pressure. By W. RAMSAY, Ph.D., and SYDNEY YOUNG, D.Sc.**

IN Wiedemann's *Annalen*, vol. xxviii. p. 400, W. Fischer has published a paper on the above subject. After stating, in the course of a historical sketch, that no experimental work with a view to decide whether the vapour-pressure of a solid is identical with that of its liquid at the same temperature below the melting-point of the solid has been carried out since Regnault's time, he corrects himself in a footnote in which he refers to a paper published by us on this subject in the *Philosophical Transactions* in 1884. He there states, however, that he gained a prize through some work on the vapour-pressures of water and of benzene in 1883. Now it is generally understood that priority is determined, not by the date at which work is done, but by the date of publication; and as Fischer's work was not made public until July 1886, there can be no question of priority between us. But in the short sentence on our work he commits a grave error in stating that our work was merely qualitative; on the contrary, it was rigorously quantitative; and we then showed that the numbers calculated for the vapour-pressure of ice, using as data an extension of the vapour-pressures of water below 0° , extrapolated from Regnault's measurements above 0° , his determinations of the heat of vaporization of water and fusion of ice, and of the specific heat of ice, agreed closely with those found by us. As regards benzene, however, our work had no pretence to be quantitative.

* Read December 11, 1886.

Fischer's experiments were made by a process identical in principle with that employed by Regnault. The water, or benzene, on which he experimented, was introduced, by a process well devised for excluding air, into a vacuum connected with a gauge, and on altering the temperature of the liquid, alteration of pressure was noted and registered. His experimental results are very regular, and, so far as water and ice are concerned, confirm ours, and agree well with theory.

But although Fischer's experimental results with benzene are equally regular, yet they present certain anomalies which are difficult to explain. From his results he calculated constants—one series to represent the relations of the pressure of vapour in contact with liquid, and the other to represent similar relations for vapour in contact with solid, employing formulæ of the general form $p = a + bt + ct^2$. From the numbers calculated by means of these constants, he concluded that *the vapour-pressure of liquid benzene is not identical with that of solid benzene at the melting-point of the solid*. This conclusion is evidently opposed to the second law of thermodynamics; and, if it had not been apparently supported by Fischer's really excellent experimental measurements, might have been dismissed at once as absurd. But on revising Fischer's results, we find that the constants employed by him, if used to calculate the vapour-pressures of the solid at low temperatures, give results which are by no means in accordance with his measurements. Indeed at -8° the calculated pressure is 13.51 millim., whereas Fischer found 14.2 millim.; and it is evident, from a graphic representation of his results, that the divergence would increase at lower temperatures. Now it is known that the relations of pressure to temperature are better expressed by means of a formula of the type suggested by Biot, $\log p = a + ba^t + cb^t$; or, for a small range of pressure, by the simpler form $\log p = a + ba^t$. On calculating constants from Fischer's results by means of this formula, we found that while a curve was obtained agreeing better with his experimental results, the anomaly which he supposed (*viz.* want of coincidence at the melting-point) no longer existed. As in the case of water and ice, solid and liquid acetic acid, solid and liquid bromine, and solid and liquid iodine, which have formed the

subject of our experiments (Phil. Trans. 1884, p. 461, and Trans. Chem. Soc. 1886, p. 453), solid and liquid benzene exert the same vapour-pressure at the melting-point.

For recalculating Fischer's results, his pressures at -7° , -2° , and $+3^{\circ}$ were taken. At -7° , the vapour-pressure of the solid determined by him was 15.273 millim.; at -2° , 21.679 millim.; and at $+3^{\circ}$, 30.324 millim. The constants are $a=4.81664$; $\log b=0.5602315$; $\log \alpha=1.99628446$; b is negative; and t = temperature centigrade $+7$. For the vapour-pressures of liquid benzene, Fischer's results agree well with his formula. As data are in existence whereby the vapour-pressures of solid benzene can be calculated for a short interval of temperature below the melting-point, provided those of the liquid are accurately known, it was deemed advisable to check Fischer's results with liquid benzene by the dynamical method already described by us (Trans. Chem. Soc. 1885, p. 42).

This was accordingly done. A large quantity of commercial benzene was distilled, and the first half, boiling within five degrees, was frozen twice, the liquid portion being poured off each time. The solid portion was then shaken repeatedly with sulphuric acid until the acid was no longer coloured, thiophene being thus removed. The remainder was shaken with water and dried, and then fractionated until a product of constant boiling-point was obtained. The actual boiling-point, at a pressure of 753.4 millim., was $79^{\circ}.9$.

The following determinations with the liquid were made. At the lower temperature, thermometers graduated in $\frac{1}{10}$ divisions were employed. The zero-points of these thermometers, and the apparent lowering of temperature consequent on reduction of pressure, were determined, and corresponding corrections introduced. For higher temperatures (above 50°), a thermometer previously used for determination of vapour-pressures, and of which the corrections had been thoroughly investigated, was used. The three $\frac{1}{10}$ thermometers we shall name A, B, and C; the one used at higher temperatures, D.

Liquid Benzene.

Series I.		Series II.		Series III.	
Temp.	Pressure.	Temp.	Pressure.	Temp.	Pressure.
	millim.		millim.		millim.
A. -1.84°	24.0	A. 5.30°	35.05	D. 31.4°	125.85
-0.97	25.1	6.25	37.2	33.6	137.9
-0.02	26.8	7.03	38.45	35.5	150.8
$+0.98$	27.9	7.47	39.85	44.7	221.0
1.23	28.15	7.98	40.9	50.1	267.0
2.03	29.15	8.37	41.8	51.85	287.4
3.08	30.95	9.21	43.0	54.25	313.9
3.13	31.15	10.44	47.25	56.4	339.5
4.84	34.7	12.51	51.65	58.8	371.4
5.06	35.2	14.66	57.8	61.2	402.7
5.31	35.4	16.85	64.5	63.6	440.0
5.48	35.9	18.72	70.75	67.2	497.6
5.74	36.15	20.54	77.25	69.25	534.6
B. 5.63	36.15	22.97	85.5	71.85	582.0
A. 8.98	42.8	25.37	96.4	74.1	630.3
B. 8.77	42.8	28.15	109.35	76.65	684.6
		30.87	123.45	79.6	743.1
		38.78	173.0		
		39.35	174.95		
		41.41	190.8		
		43.71	209.6		
		45.97	230.0		
		47.94	249.1		
		49.36	263.3		

A curve was drawn to represent these relations ; and from it three points were chosen, viz. 0° , 40° , and 80° ; the corresponding pressures are: 0° , 26.54 millim.; 40° , 180.2 millim.; and 80° , 755.0 millim. The constants for the formula $\log p = a + b\alpha^t$ are

$$a = 4.72452; \log b = 0.5185950; \log \alpha = 1.996847125;$$

b is negative.

The following Table shows the calculated vapour-pressure for each 10° .

Temperature.	Pressure.	Temperature.	Pressure.
	millim.		millim.
-10°	14.97	40°	180.20
0	26.54	50	268.30
10	45.19	60	388.51
20	74.13	70	548.16
30	117.45	80	755.0

These numbers agree fairly well with Regnault's results at and above 10°. Below that temperature his constants are calculated from the vapour-pressures of the solid as well as of the liquid, and of course are therefore incorrect.

We give a Table of comparison of our calculated results, with those calculated by Fischer, between 0° and 6°.

Temp.	Pressure.		Temp.	Pressure.	
	F.	R. and Y.		F.	R. and Y.
0	millim. 26.40	millim. 26.54	4	millim. 32.84	millim. 32.99
1	27.87	28.04	5	34.68	34.80
2	29.43	29.61	6	36.60	36.69
3	31.10	31.26			

It will be seen that the agreement is a very close one; and as our determinations were made by the dynamical method, while Fischer's were obtained statically, there is a strong presumption that the substance in both cases was pure.

In order to calculate the vapour-pressures of solid from those of liquid benzene, the following formula was employed:—

$$\text{Vapour-pressure of solid at } (t-1) = P_t - (P'_t - P'_{t-1}) \left(\frac{V_{t-\frac{1}{2}} + F_{t-\frac{1}{2}}}{V_{t-1}} \right);$$

P = vapour-pressure of solid ;

P' = vapour-pressure of liquid ;

V = heat of vaporization of liquid ;

F = heat of fusion of solid ;

t = temperature of solid and liquid.

It is therefore necessary to know the heat of vaporization of liquid benzene at different temperatures, and the heat of fusion of solid benzene; and in order to calculate these, the specific heats of liquid and of solid benzene. The following determinations are available:—

1. *Heat of Vaporization of Liquid Benzene.*—Regnault (*Mémoires de l'Institut*, xxvi. p. 881) has determined the total heats of vaporization of benzene at different temperatures, while Schiff (*Annalen*, ccxxxiv. p. 344) has made a single

determination at the boiling-point. Regnault's formula is

$$H = a + bt + ct^2,$$

where $a=109$, $b=0.24429$, and $c=-0.0001315$. Schiff's single determination at the boiling-point, $80^{\circ}.35$, at a pressure of 765.1 millim., is 93.4 to 93.5 calories.

2. *Heats of Fusion of Solid Benzene.*—Peterson and Widmann (*J. prakt. Chem.* xxiv. p. 129) give the number 29.09 calories; and Fischer (*loc. cit.*) found 30.085 calories.

3. *Specific Heat of Liquid Benzene.*—This has been often observed. But isolated observations are for our purpose comparatively valueless, for the specific heat varies with the temperature. Formulæ are given only by Schüller (*Pogg. Ann., Ergänzungs-Band*, p. 5), and by Schiff (*loc. cit.*). We have used Schiff's formula. It is, specific heat $= a + bt$; where $a=0.3834$ and $b=0.001043$. Between narrow limits of temperature this may be accepted as sufficiently correct.

4. *Specific Heat of Solid Benzene.*—Fischer (*loc. cit.*) gives 0.319.

Calculating the heat of vaporization at the boiling-point under normal pressure from Regnault's total heats of volatilization and Schiff's specific heat of liquid benzene, the number 93.67 is obtained; while Schiff found by direct measurement 93.4 to 93.5. This is a strong presumption in favour of the correctness of the data.

The mean of the two determinations of the heats of fusion of benzene was taken.

The following Table summarizes the data for calculating the vapour-pressures of the solid. But this calculation involves the assumption that the heat evolved on solidification at any low temperature is equal to that evolved at the ordinary melting-point, minus the product of the specific heat of the solid into the difference of temperature; and that the specific volume of the vapour in contact with solid is equal to that of vapour in contact with liquid. It is certain that neither of these assumptions is true; hence it is not legitimate to calculate the vapour-pressures of the solid from those of the liquid. Still, for some degrees below the melting-point, the error involved in these assumptions is probably not very great.

Temp.	P'.	$P'_t - P'_{t-1}$.	$V_{t-\frac{1}{2}}$.	$F_{t-\frac{1}{2}}$.	$\frac{V_{t-\frac{1}{2}} + F_{t-\frac{1}{2}}}{V_{t-\frac{1}{2}}}$.	$P_t - P_{t-1}$.	P.
	mm.	mm.	calor.	calor.			mm.
5.58	35.89	1.86	108.3	29.4	1.271	2.365	35.89
4.58	34.03	1.78	108.4	29.1	1.268	2.258	33.52
3.58	32.25	1.69	108.5	28.8	1.265	2.139	31.27
2.58	30.56	1.62	108.6	28.5	1.262	2.045	29.13
1.58	28.94	1.54	108.8	28.2	1.260	1.941	27.08
0.58	27.40						25.14

The vapour-pressures of solid benzene, determined by our method, are given in the following Table. As Fischer's method was statical, while ours is dynamical, a comparison of the results of both is therefore given.

Pressure.	Temp.	Pressure.	Temp.	Pressure.	Temp.
SERIES I.		SERIES I. (cont.).		SERIES II. (cont.).	
millim.		millim.		millim.	
35.9	+5.43	21.8	-0.95	14.0	-7.02
35.4	5.21	21.4	-1.12	13.25	-7.37
35.2	5.07	21.0	-1.45	13.10	-7.53
34.7	4.89	19.95	-2.00	12.80	-8.42
31.15	3.62	19.0	-2.54	11.6	-8.77
30.95	3.32	19.0	-2.97	11.2	-9.90
29.15	2.70	18.05	-2.98	10.2	-9.72
29.15	2.75	16.3	-4.63	10.1	-10.54
28.7	2.41	16.25	-5.45	9.95	-11.03
28.15	2.21	13.8	-6.43	7.35	-14.12
27.9	1.99	12.35	-8.07		
26.8	1.29			SERIES III.	
26.7	1.33			14.25	-7.01
26.65	1.33	SERIES II.		14.0	-7.70
25.9	0.85	21.9	-1.26	10.4	-11.2
25.1	0.64	21.1	-1.60	10.2	-11.0
24.45	0.23	20.05	-2.61	10.2	-10.63
24.0	0.0	18.75	-3.08	9.95	-11.62
23.9	-0.03	17.15	-4.38	9.3	-11.92
22.75	-0.78	15.2	-5.78	9.3	-11.3
22.2	-0.91	15.0	-6.02	8.7	-12.12

For the first two series thermometers A and B were used ; for Series III. another thermometer, D ; and it will be seen that its readings confirm those of the other two. The individual results are not so concordant as Fischer's. The reason is that the volatilization of solid benzene is so quick as to make it difficult to obtain an accurate reading before

the solid has volatilized and partially exposed the thermometer-bulb. These numbers, near the melting-point, show close concordance with those of Fischer, but at lower temperatures they show signs of divergence. For example at -8.42 , if Biot's formula be applied, the difference between Fischer's results and ours amounts to 1.39 millim.; and it is evident, from the graphical representation, that the difference would be an increasing one. It is to be noted that Biot's formula agrees with Fischer's own results much better than the formula employed by him.

Our results were plotted on curve paper, and the constants for a formula calculated from points on a curve drawn to pass well through them. The constants for the formula $\log p = a + ba^t$ are

$$a = 4.82602; \log b = 0.5784772; \log a = 1.9959086;$$

$t = t^\circ \text{Centig.} + 10$; b is negative.

The following Table comprises (A) the values calculated from the vapour-pressures of the liquid; (B) those calculated from the formula and constants given by Fischer; (C) those calculated from Fischer's results by Biot's formula; and (D) those calculated from our results by Biot's formula.

Temperature.	A.	B.	C.	D.
	millim.	millim.	millim.	millim.
5.58	35.89	35.62	35.85	35.86
4.58	33.52	33.36	33.62	33.39
3.58	31.27	31.42	31.50	31.07
2.58	29.13	29.54	29.50	28.90
1.58	27.08	27.73	27.61	26.85
0.58	25.14	25.97	25.82	24.94
-0.42	24.28	24.14	23.14
1.42	22.65	22.55	21.46
2.42	21.09	21.06	19.89
3.42	19.59	19.66	18.41
4.42	18.15	18.33	17.04
5.42	16.77	17.09	15.75
6.42	15.45	15.92	14.56
7.42	14.20	14.82	13.44
8.42	13.01	13.79	12.40
9.42	11.89	12.82	11.43
10.42	10.53
11.42	9.69
12.42	8.91

It is possible to calculate the heat of fusion of solid benzene,

from the value of $\frac{dp}{dt}$ at the melting-point, from the equation

$$\frac{dp}{dt}(\text{solid}) - \frac{p}{dt}(\text{liquid}) = \frac{V + F}{V},$$

where V is the heat of vaporization at the melting-point, and F the heat of fusion.

With Fischer's formula and constants . $F = 6.29$ calories

With Fischer's results and Biot's formula. $F = 21.1$ „

With our results and Biot's formula . . $F = 35.4$ „

The number found by Fischer . . . $F = 30.085$ „

The number found by Peterson and
Widman $F = 29.09$ „

From these numbers it is evident that, although our constants are not perfectly correct, yet they agree better with experimental evidence than those of Fischer.

XXII. *On the Nature of Liquids, as shown by a Study of the Thermal Properties of Stable and Dissociable Bodies.* By WILLIAM RAMSAY, *Ph.D.*, and SYDNEY YOUNG, *D.Sc.**

THE fundamental concept of Chemistry, as well as of Physics, is the molecular and atomic constitution of matter. This concept serves to represent to the chemist the definite composition of compounds, and, to some degree, the nature of isomerism, while all attempts to realize and explain the progress of chemical change depend on its adoption. This concept also furnishes to the physicist the means of conceiving the relations of heat, light, magnetism and electricity to matter ; and where the action of one of these agents involves not merely a change in the form, but also in the nature of the matter, the problem becomes of deep interest to both chemist and physicist. The action of heat on matter, from the physical side, involves an increased molecular motion, tending to separate individual molecules from each other, on the one hand ; or, on the other, if this separation be opposed by confining walls, to increase the momentum and

* Read December 11, 1886.

number of impacts on those walls, and therefore to raise the pressure. But this increased molecular motion is accompanied by greater internal vibration, which eventually leads, in almost all cases, to a simplification or rearrangement of the molecules, involving chemical change. When increased molecular motion is imparted to gases at temperatures much above their points of condensation, and at moderate pressures, the problem is a comparatively simple one; and has been solved with great success by Clausius, Maxwell, Thomson, and others, from the physical side, and from the chemical side by Pfaundler, Naumann, and Willard Gibbs. But near their condensing points, and also at high pressures, Boyle's and Gay-Lussac's laws no longer hold, owing partly no doubt to the mutual attraction of the molecules, and also to the fact that the absolute size of the molecules is no longer insignificant relatively to the space which they occupy. Both these causes of deviation may be relegated to the class "physical," inasmuch as the mutual attraction alluded to is not confined to any small number of molecules, but is exercised by each molecule on all its neighbours, and limited in absolute amount only by the relative masses of the attracting molecules and by their distances from each other. But it is also conceivable that this attraction may be wholly or in part of a chemical nature, tending towards the formation of complex molecules, resulting from combination of two or more simple molecules. Now as this deviation from the simple gaseous laws occurs both with what are commonly termed "stable" and with "dissociable" substances, it is of importance to enquire whether the abnormality of the vapour-density of stable substances is at all due to chemical association of molecules; and how much of the abnormality of dissociable substances is to be ascribed to purely physical attraction of the molecules for each other, due to mere propinquity.

At any temperature below the critical one, when the volume of gas is decreased, pressure rises until a certain maximum is attained, when it becomes constant, and change of state occurs. It is conceivable, on the one hand, that the liquid condition is a purely physical one, and that a liquid consists of molecules similar in all respects to those of its gas, but, owing to their closer proximity, exhibiting that form of at-

traction which is known as cohesion. And on the other hand, it has been advanced by Naumann and others that the gaseous molecules, in changing to liquid, form molecular groups of definite complexity, exercising cohesive attraction on each other; and, according to this view, the problem is both a physical and a chemical one. According to the first view, if heat be imparted to a liquid, work is done in expansion against pressure, and in overcoming cohesion; and, according to the second view, additional work is done in dissociating the complex molecules into their simpler constituents, and in imparting increased velocity and internal motion to those constituent molecules (see "Evaporation and Dissociation," part i., Trans. Roy. Soc. 1886, Part I.).

When a substance, such as chloral hydrate or ammonium chloride, passes from the solid or liquid into the gaseous state, the physical change is obviously accompanied by a chemical one, for dissociation into simpler molecules occurs. There is an obvious analogy between evaporation and such cases of dissociation; and we have recently undertaken experimental work to test whether this analogy is a real one.

In part i. of this series of papers the phenomena attending the volatilization of such solids as dissociate wholly or partially on their passage from the solid to the gaseous state have been studied. There are two ways of measuring the vapour-pressure of a stable substance, which have been termed by Regnault the statical and the dynamical respectively. The first consists in measuring the pressure exercised by the vapour of the substance kept at a uniform temperature; and the second in measuring the highest temperature attainable by the substance at given pressures, when evaporation freely takes place. It has been shown by Regnault, and by numerous other observers, that these methods give identical results with liquids, and by ourselves with solids (Trans. Roy. Soc. Part I. 1884, p. 37). But in the case of the majority of the dissociable bodies examined, the results of the two methods were not identical; indeed, in many cases in which dissociation is complete, or nearly so, the temperature of volatilization is independent of pressure. With nitrogen peroxide, acetic acid, and ammonium chloride, however, the two methods gave identical results. This method, therefore, cannot be

regarded as a means of deciding the question of the analogy between evaporation and dissociation, unless, indeed, two kinds of chemical combination be conceived, one of which may be termed "molecular combination" as distinguished from "atomic combination."

In parts ii. and iii. the thermal behaviour of stable liquids has been investigated, as exemplified by alcohol and ether. For a complete account of these researches reference must be made to the original papers (Trans. Roy. Soc. 1886, Part I.*). We are here concerned chiefly with the densities of the saturated vapours, and with the heats of vaporization. We found, with alcohol, that the density of the saturated vapour was normal at temperatures below 40° or 50° , and remained normal down to a temperature of 13° , the lowest temperature at which observations could be made. With ether the vapour-density was approaching normality at 13° , and from the form of the curve would have doubtless become normal at a lower temperature. In both cases, with increase of temperature and corresponding increase of pressure, the density of the saturated vapour increased towards the critical point with great rapidity, until at the critical point the weight of unit volume of the saturated vapour was equal to that of the liquid.

At the critical point the heat of vaporization of a stable liquid is theoretically zero; below that temperature we found it to increase with alcohol and with ether as the temperature fell; with ether the increase was found to be continuous to the lowest observed temperature 13° ; whereas, with alcohol, it becomes practically constant below about 20° . Our calculated numbers correspond well with direct measurements by various observers at the boiling-points under atmospheric pressure.

With acetic acid the results were very different. With rise of temperature above 150° the density of the saturated vapour increased, as with other liquids; but below that temperature (at which the vapour-density was 50.06, the calculated density being 30) the vapour-density, instead of continuing to fall, rose more and more rapidly with fall of temperature, until at 20° the vapour-density was approximately 59, and apparently, from the form of the curve, was

* The constants for ether will be published shortly.

continuing to rise more and more rapidly, with fall of temperature (see figs. 2 & 3). It may be mentioned that direct observations by Bineau at 20° give nearly the same value.

Fig. 1.

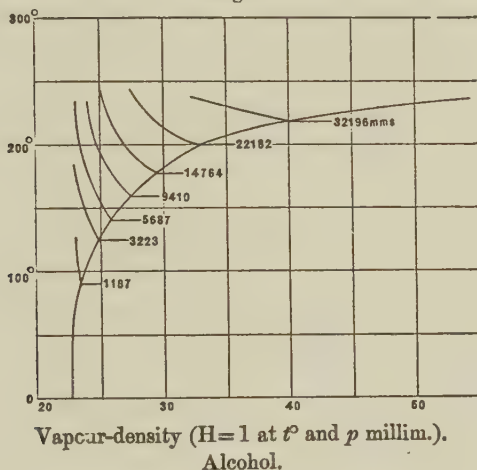
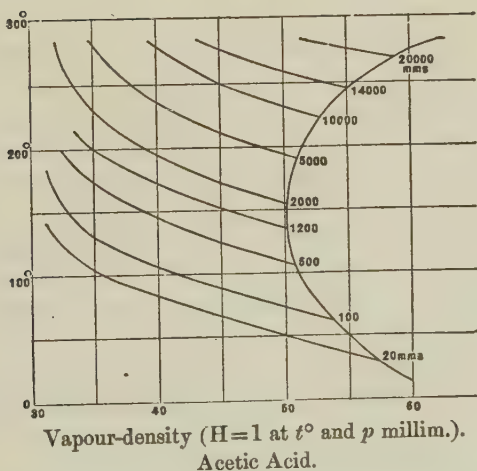


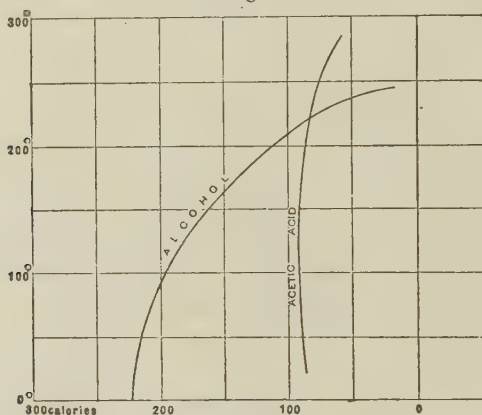
Fig. 2.



The curve representing heats of vaporization of acetic acid at various temperatures also differs entirely in form from those of alcohol and ether, for it exhibits a maximum at 110° , and

decreases both with rise, and with fall of temperature. It is difficult to draw any conclusion from a comparison of our measurements of this quantity at the boiling-point under atmospheric pressure with those of other observers; but it may be stated that our result differs far less from the observation of Favre and Silbermann than theirs does from that of Berthelot (see fig. 3).

Fig. 3.



It appears to us that these results negative the "chemical" explanation of the constitution of liquids, or, to confine ourselves to known cases, of the liquids alcohol and ether. The molecules of these liquids cannot, we think, be regarded as complex, consisting of gaseous molecules in chemical combination with each other, as, for example, $n(\text{C}_2\text{H}_6\text{O})$, where n is any definite number. We believe, rather, that the physical explanation of the nature of liquids is the correct one, and that the difference between liquids and gases lies merely in the relative proximity of their molecules.

The chief argument for this view is that it is difficult to conceive that the rise of vapour-density of acetic acid, both at high and at low temperatures, can be due to the same cause, under conditions so radically different; for at high temperatures we have conditions unfavourable to chemical combination, but owing to the necessarily high pressure, the molecules are in close proximity; whereas, at low temperatures, the conditions are favourable to chemical combination,

while the molecules, owing to the corresponding low pressures, are very far apart. Now we have shown that, with alcohol and with ether, a rise of density does not accompany fall of temperature ; indeed, the saturated vapour of alcohol, at low temperatures, obeys the laws of Boyle and Gay-Lussac; while the rise of vapour-density at high temperatures is common to all bodies. But with acetic acid, the lower the temperature the higher the density of its saturated vapour—a fact which indicates the formation of complex molecules ; at high temperatures, however, it forms no exception in behaviour to ordinary liquids.

We have shown that with stable substances there is proof of the absence of complex molecules in their vapours ; but it might be asserted that in the passage from the gaseous to the liquid state, combination might occur. That this cannot be the case, is evident from a consideration of the behaviour of liquids near their critical point. For the specific volumes of liquid and gas just below the critical point are nearly-equal ; and were the liquid to consist of congeries of gaseous molecules, there would necessarily be fewer molecules in unit volume of the liquid than in unit volume of the gas—an improbable conception.

It is impossible to decide from our experiments whether the higher limit of vapour-density of acetic acid is 60 ; and the difficulty of measuring small pressures with sufficient accuracy renders an answer to this question apparently impossible ; but it is a remarkable circumstance that our observations, as well as those of Bineau, should so closely approximate to this limit. Although the curves representing the density of the saturated vapour in figs. 1 and 2 apparently point to a vapour-density greater than 60, yet a trend in the curve is not impossible ; and it is conceivable that at lower temperatures than those represented, the density might remain normal for $C_4H_8O_4$.

If there is a definite limit to the vapour-density of acetic acid, then the following considerations will hold. It has been pointed out in our paper on acetic acid, that condensation took place before pressure ceased to rise ; and the same phenomenon was observed with chloral ethyl-alcoholate, where dissociation is known to occur. Now with alcohol and with

ether absolutely no sign of this behaviour was observable; condensation occurred the moment the vapour-pressure was reached, but not till then. This behaviour corresponds to that of a mixture. If an indifferent gas, to take an extreme instance, is compressed along with the vapour of a condensable liquid, pressure continues to rise after condensation has commenced, until the gas, if possible, has been dissolved, or has itself condensed. On the other hand, if a small quantity of liquid of high boiling-point be present along with a large quantity of liquid of low boiling-point, the liquid of higher boiling-point separates out first, on reduction of volume, while pressure continues to rise. This was indeed noticed with an impure sample of ether; and the absence of this behaviour affords proof of the homogeneity of a liquid.

Supposing the vapour of acetic acid to consist of molecules of two different degrees of complexity, it is probable that the more complex would be first condensed, and that pressure would rise until the less complex molecules had also condensed. This was in fact observed. But below a certain temperature the substance would consist almost wholly of more complex molecules, and the phenomenon would then be less visible. This is indeed the case with the isothermals at 50° and at $78^{\circ}4$. At higher temperatures the phenomenon becomes evident. That this behaviour is not the effect of impurity has been proved by the fact that the vapour-pressures at low temperatures, measured by the statical and by the dynamical methods, were identical.

Formulae representing the dependence of dissociation on pressure and temperature have been proposed from thermodynamical considerations by Prof. Willard Gibbs*. The formula is for acetic acid

$$\log \frac{2.073(D-2.073)}{(4.146-D)^2} = \frac{3520}{t_c + 273} + \log p - 11.349.$$

The numbers 2.073 and 4.146 are the densities referred to air of the molecules $C_2H_4O_2$ and $C_4H_8O_4$ respectively; D is the observed density; and 3520 and 11.349 are constants deduced from the determinations of Cahours and Bineau. This formula, of which the constants in its author's opinion

* American Journal of Science and Arts, 1879, p. 277.

can claim only approximate correctness, is quite inadequate to represent actual facts at high temperatures and high pressures where cohesion becomes marked. For example, it gives at a temperature of 280° for the density of the saturated vapour 35.13 instead of the observed number 62.62.

If our opinion be correct, and if the abnormal density of saturated vapours and of vapours near their saturation-points and also above their critical points, at high pressures, of stable substances, be due to mere molecular proximity, and not to any form of molecular combination; then a dissociating substance must exhibit a vapour-density which may be partly due to this cause. With such a substance as ammonium chloride, which, we have shown, is almost completely dissociated at 280° , the products of dissociation (hydrogen chloride and ammonia) are under such conditions of temperature and pressure that they would probably behave as perfect gases; the relatively few molecules of ammonium chloride which remain undecomposed in the gaseous state are under such low pressure, that their density is probably normal for the formula NH_4Cl ; and in this case it is probable that the chemical factor alone determines the vapour-density. But with acetic acid the increase of density above 150° is evidently wholly due to the physical cause; while the abnormality is partly due to a physical, partly to a chemical, cause. It is, however, impossible in this case to ascertain at what temperature the physical cause begins to operate. It is evidently to be wished that, from a study of the behaviour of stable substances, some general law could be discovered which would embrace all instances of physical abnormality; and many attempts have been made in this direction, but as yet with only partial success. Willard Gibbs, on the other hand, has attacked the problem from the chemical side; and we have shown that his formula ceases to apply when the physical change becomes predominant.

Messrs. E. and L. Natanson* have recently published a research on the vapour-densities of nitric peroxide (N_2O_4 or NO_2), which, taken in conjunction with experiments of ours on the vapour-pressures of that body (Phil. Trans. 1886, Part I.), affords a striking confirmation of the correctness of

* Wiedemann's *Anna'en*, 1886, p. 606.

our views. They give an isolated observation at $-12^{\circ}6$; and isothermals at 0° , 21° , $49^{\circ}7$, $73^{\circ}7$, $99^{\circ}8$, $129^{\circ}9$, and $151^{\circ}4$. The limit of pressure was 800 millim. Now the boiling-point of nitric peroxide is, from our measurements, $21^{\circ}8$; hence the densities of the saturated vapour are deducible from only the first three of the Messrs. Natansons' isothermals. We have plotted their results on curve-paper; this has shown us the regularity and trustworthiness of their observations; and by continuing the curves in the direction in which they run until they intersect the straight lines denoting vapour-pressures at the temperatures at which their measurements were made (using for this purpose the vapour-pressures determined by us), the density of the saturated vapour is determined with but small error.

The Natansons' numbers are as follows:—

Temp.	Pressure.	Density.	Temp.	Pressure.	Density.
$-12^{\circ}6$	millim. 115.4	millim. 52.54	21°	millim. 491.60	millim. 38.74
$0^{\circ}0$	37.96	35.84		516.96	39.01
	86.57	38.59		553.50	39.15
	172.48	40.71		639.17	39.64
	250.66	41.90			

At $-12^{\circ}6$ the vapour-pressure of nitric peroxide is 125 mm. The density of the saturated vapour must therefore be a little above 52.54. Now the theoretical density of N_2O_4 is 46. It may be that the higher density is due to experimental error; but from graphic representation of the Natansons' results this appears improbable. If the measurement is correct, it would imply that the chemical combination of molecules of NO_2 is not complete when the molecular complexity is represented by the formula $(NO_2)_2$, but may extend to $(NO_2)_3$, or even further. At 0° the vapour-pressure is 255 millim.; again the density found by the Natansons must be nearly that of saturation. At 21° the vapour-pressure is about 700 millim.; and a prolongation of the curve constructed from the above numbers would cut the line representing the large alteration of volume with no rise of vapour-pressure at a vapour-density of about 40. It is evident, then, that with nitric peroxide, as

with acetic acid, the density of the saturated vapour rises with fall of pressure and temperature. Now it is known that nitric peroxide dissociates, for the physical properties (colour, &c.) change, on change of $(\text{NO}_2)_n$ into $n(\text{NO}_2)$; and the similarity of behaviour between nitric peroxide and acetic acid renders the dissociation of acetic acid from $(\text{C}_2\text{H}_4\text{O}_2)_n$ into $n(\text{C}_2\text{H}_4\text{O}_2)$ no longer conjectural.

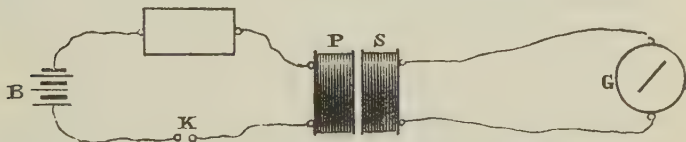
If it be granted that our thesis is proved, that the molecules of stable liquids are not more complex than those of their gases, it follows that the difference between liquids and gases is one of degree, not of kind; is quantitative, and not qualitative.

University College, Bristol,
November 18, 1886.

XXIII. *Note on a Method of Determining Coefficients of Mutual Induction.* By G. CAREY FOSTER, F.R.S.*

THE determination in absolute measure of the coefficient of mutual induction of two electric circuits by the ordinary method founded on the throw of a ballistic galvanometer is a somewhat complicated matter, necessarily occupying a good deal of time. But the process may be greatly simplified if we have available a condenser of which the capacity is accurately known. For instance, if P and S are two coils whose coefficient of mutual induction is required, let them first be joined up, as indicated in fig. 1—P in a primary

Fig. 1.



circuit containing a battery, B, and make-and-break key, K; and S in a secondary circuit of total resistance r including a ballistic galvanometer G. Then, on closing or opening the key K, a momentary deflection of the galvanometer will occur in consequence of its being traversed by a quantity

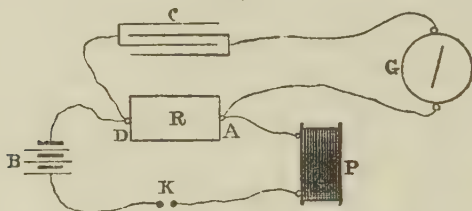
* Read November 27, 1886.

of electricity Q given by

$$Q = M \frac{\gamma}{r},$$

where M is the coefficient of mutual induction between P and S , and γ the strength of the primary current. Next, leaving the primary circuit unaltered, let connections be made as indicated by fig. 2, where C is a condenser of known capacity

Fig. 2.



C , and A and D are two points in the primary circuit separated by a resistance R . Then, on making or breaking contact at K , the galvanometer is traversed by a quantity Q' , such that

$$Q' = \gamma CR.$$

If the points A and D are found by trial so that the deflection of the galvanometer is the same in both cases, we have

$$M = CRr.$$

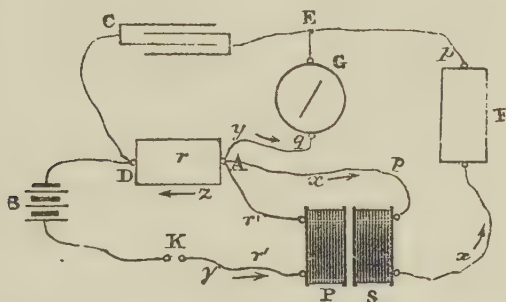
This mode of working, however, has the obvious defect that the result, as stated, implies that the current in the battery-circuit is of exactly the same strength during each part of the experiment. As this cannot be looked for, it would be needful to include a measuring galvanometer in the battery-circuit, so as to take account of the variation of the current. The required coefficient is then obtained in the form

$$M = \frac{\gamma_2}{\gamma_1} CRr.$$

But, instead of making two separate experiments, as above, it is simpler to adopt an arrangement of apparatus which is very nearly a combination of the two arrangements just described. A single experiment then takes the place of two, and, instead of having to reproduce a particular deflection of the galvanometer, we have to adjust a resistance so as to prevent deflection. The connections will be understood by reference to fig. 3, where, so far as the reference-letters used in the previous figures recur, they have the meanings already

given to them. The observation consists in adjusting a set of resistance-coils at F, between the galvanometer and the

Fig. 3.



coil *s*, until there is no throw of the galvanometer on making or breaking contact at *K*.

Let the resistances *ASFE*, *AGE*, *AD* be represented by *p*, *q*, *r* respectively, and the corresponding currents by *x*, *y*, *z*. Further, let the current in the battery and primary coil be denoted by *γ*, the coefficient of self-induction of the coil *s* by *L*, and the potentials at the points *A* and *E* by *A* and *E* respectively.

When the battery-current has attained its steady value, it is evident that the currents *x* and *y* will both be nothing, and therefore that *A*=*E*, and that the charge of the condenser will be *Cγr*. But if there has been no throw of the galvanometer-needle, the average value of the current *y* during the whole time of establishment of the battery-current has been =0. Consequently, the total quantity conveyed by the current *x* has been equal to the charge of the condenser, or

$$\int_0^x x dt = C\gamma r.$$

But, if the average strength of the current *y*=0, the average difference of potentials *A*−*E*=0, and the effective electromotive force in the conductor *p* is that due to the mutual induction of the coils *P* and *s* (for the integral value of the electromotive force of self-induction must vanish). Hence

$$\int_0^x x dt = \frac{M}{p} \int_0^x \frac{d\gamma}{dt} dt = C\gamma r,$$

or

$$M = Cpr.$$

In order that the galvanometer-current y may be zero at every instant, as well as on the average, during the establishment of the primary current, it is essential that the coefficient of self-induction, L , of the coil s should be equal to the coefficient of mutual induction M . This may be proved as follows:—Since, in the case supposed, we have always $A-E=0$, we may write

$$px + L \frac{dx}{dt} - M \frac{d\gamma}{dt} = 0.$$

But $x = \gamma - z$ (since $y = 0$, always), consequently the instantaneous value of the current x is

$$x = \frac{1}{p} \left[(M-L) \frac{d\gamma}{dt} + L \frac{dz}{dt} \right],$$

and the simultaneous charge of the condenser is

$$C \, rz = \int_0^t x \, dt = \frac{1}{p} \left[(M-L) \gamma + Lz \right],$$

or

$$(M-L)\gamma = (Cpr - L)z.$$

But, since it has been already proved that $M = Cpr$ when the average current through the galvanometer vanishes, the last equation becomes

$$(M-L)\gamma = (M-L)z,$$

which requires either (1) that $\gamma = z$ (that is, that the primary current has become steady), or (2) that $M-L=0$.

When the last-mentioned condition is satisfied, we ought to be able to replace the galvanometer between the points A and E by a telephone, and to employ a rapid make-and-break*.

By using a pair of coils of known and invariable coefficient of mutual induction, the experimental process described above may afford a ready way of determining the capacity of a condenser; but for this purpose the method does not appear to possess any advantage over the well-known methods of De Sauty and Sir William Thomson for determining the capacity of condensers by comparison with a known standard,

* Experiments in this direction, made since the reading of this paper before the Physical Society, by Mr. F. Womack, have not yet led to a fully satisfactory result.

more especially as a known condenser is more frequently at hand than a pair of coils of known coefficient *.

The limit of accuracy attainable in measurements by the method here described depends essentially upon the sensitiveness of the galvanometer employed; but with a given galvanometer, the method is susceptible of various degrees of accuracy according to the relative values given to the two variable resistances p and r .

These should be so adjusted that, for a given value of the difference $Cpr - M$, that is, for a given error in the adjustment of the product of the resistances p and r , the quantity of electricity traversing the galvanometer may be as great as possible. This requires that the resistances should fulfil the condition

$$\frac{p}{r} = \frac{q}{r'},$$

where r' is the resistance between A and D through the primary coil and battery. Let X and Y stand for the integrals $\int_0^x x dt$ and $\int_0^y y dt$ respectively, that is, for the total currents through the secondary coil and through the galvanometer. Then, equating the integral values of the electromotive force between A and E derived by considering the paths AGE and AFE, respectively, we get

$$pX - qY = M \frac{E}{r + r'},$$

E being here the electromotive force of the battery B.

* Since this paper was read, Professor Ròiti, of Florence, has kindly sent me a copy of a paper communicated by him to the Royal Academy of Sciences of Turin (Memoirs of the Academy, series ii., vol. xxxviii.), in which he describes a method, very similar to that indicated in the text, for the absolute measurement of the capacity of condensers, using for the purpose a pair of coils whose coefficient of mutual induction is accurately known from their dimensions and relative positions. The chief difference between Professor Ròiti's arrangement and that given in this paper is that, instead of inserting the galvanometer between the points marked A and E in fig. 3, he includes it in the branch, AFE, containing the secondary coil. Between A and E he places a contact-key H, which, as well as the key K, is opened and closed mechanically, the two keys being moved by cranks attached to the same axle at right angles to each other.

Again, the final charge of the condenser is

$$X + Y = CE \frac{r}{r + r'}.$$

Eliminating X from these two equations,

$$\frac{Y}{Cpr - M} = \frac{E}{(r + r')(p + q)},$$

and this is to be as great as possible.

The denominator on the right may be written

$$pr \left(1 + \frac{r'}{r} + \frac{q}{p} + \frac{qr'}{pr} \right);$$

the last term inside the bracket is the product of the second and third, and may be taken as constant, since q , the resistance of the galvanometer, and r' , the resistance of the part of the circuit containing the primary coil, are practically determined by the apparatus employed, while pr has the constant value M/C . Hence, $Y/(Cpr - M)$ is greatest when $(p + q)(r + r')$ is least, or when $p/q = r/r'$, which is identical with the condition of maximum sensibility given above*.

In conclusion, I may give a few numerical results as examples of the applicability of the method; they are derived from experiments made in the laboratory of University College, London, by Mr. F. Womack, B.Sc.

A. *Small Induction-Coil* (without iron core). Approximate dimensions:—Primary: length 11.5 cm.; mean radius 2 cm.; wire, No. 20 B.W.G.; resistance 1.65 ohm. Secondary: length 10.4 cm.; radius, inside 2.55 cm., outside 3.83 cm.; wire, No. 30 B.W.G.; resistance 194 ohms. Battery, 2 Groves. Condenser, 4.926 microfarads (by direct measurement with ballistic galvanometer). The secondary coil could slide endways while remaining coaxial with the primary. The first measurements were made with the centres of the primary and secondary as nearly coincident as possible, so as to give a maximum coefficient of induction. The following are the results obtained:—

* For the mathematical theory of the method, so far as it is given above, I am greatly indebted to my friend Dr. A. H. Fison.

<i>r.</i>	Ohms.	Product <i>pr</i> (absolute units).
	<i>p</i> (=secondary coil + resistance-box).	
15	194 + 217	6165×10^{18}
14	+ 247	6174
13	+ 282	6188
12	+ 322	6192
11	+ 367	6171
10	+ 423	6170
9	+ 490	6156
8	+ 576	6160
7	+ 688	6174
6	+ 835	6174

$$\text{Mean value of } M/C = 6172.4 \times 10^{18}$$

$$\text{Hence } M = 4.926 \times 10^{-15} \times 6172 \times 10^{18} = 3.0403 \times 10^7.$$

In these experiments the value of q/r' was about

$$135/(1.65 + .6) = 60.$$

Hence the greatest sensibility would be with $p = 60r$. This condition was nearly fulfilled with

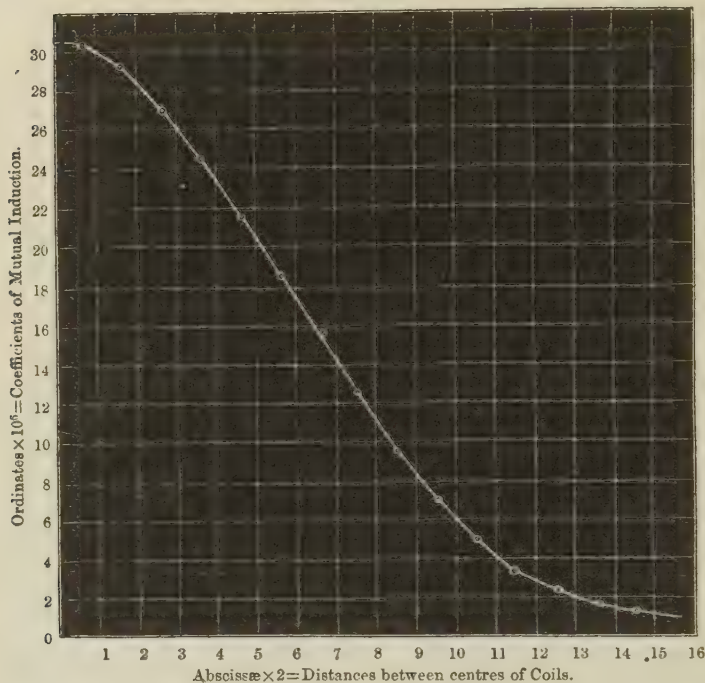
$$r = 10 \text{ and } p = 194 + 423 = 617.$$

In the same way the values of M were obtained for the same pair of coils after displacing the secondary coil endways through various distances. The following are the results obtained :—

Distance between centres of coils. centim.	Value of M .
0.55	304.0×10^5
1.55	292.4
2.55	270.5
3.55	246.4
4.55	215.9
5.55	187.8
6.55	158.4
7.55	127.2
8.55	97.3
9.55	71.1
10.55	49.7
11.55	33.0
12.55	23.3
13.55	16.5
14.55	12.35
15.55	9.48

These values are represented graphically in the curve (fig. 4),

Fig. 4.



where ordinates denote values of M and abscissæ distances between the centres of the coils.

B. *Induction-Coil*, by Apps, capable of giving a 7-inch spark in air (presented to the University College Laboratory by Mr. J. Rose-Innes, B.Sc.). Length of secondary coil 21 cm., diameter (measured outside velvet covering) 11.3 cm. Resistance of primary wire = 0.278 ohm (at $16^{\circ}5$); resistance of secondary wire 7394 ohms (at $16^{\circ}5$). Battery, 1 Grove. $q = 135.6$; $r' = 0.58$ (about). Condenser, 4.926 microfarads.

Ohms.		pr
r .	p .	(absolute units).
27	7394 + 1550	2.415×10^{23}
28	+ 1250	2.420
29	+ 940	2.417
30	+ 650	2.413
31	+ 390	2.413
32	+ 150	2.414

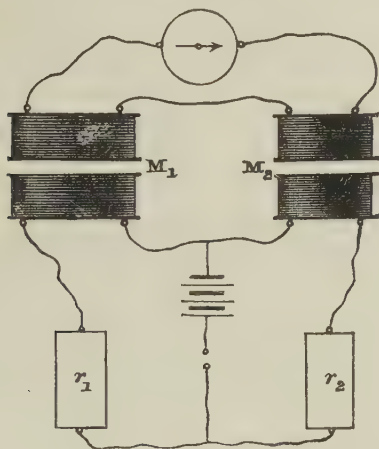
$$\text{Mean value of } M/C = 2.415 \times 10^{23}$$

Hence

$$\begin{aligned} M &= 4.926 \times 10^{-15} \times 2.415 \times 10^{23} \\ &= 1.1896 \times 10^9. \end{aligned}$$

As a further test of the accuracy of the method, the secondary wires of Apps's coil and of the small coil (A) were connected in circuit with each other and with a galvanometer, and the two primaries were connected so that the battery-current was divided between them, as shown in fig. 5. The

Fig. 5.



connections being arranged so that the induced electromotive forces opposed each other, it was possible, by the proper adjustment of the resistances of the branches containing the two primary coils, to prevent the galvanometer being deflected when the battery-circuit was made or broken. When this adjustment is made, it is evident that the ratio of the resistances of the two primary circuits is the same as the ratio of the two coefficients of mutual induction. In this way, 39.45 was obtained as the ratio of the coefficients; whereas the condenser method described above gives $\frac{118.96}{3.0403} = 39.13$, giving a difference of

about 1 per cent. As, however, the current in the primary of the smaller coil has to be about forty times as strong as that in the large coil in the comparison experiments, in order that the induced electromotive forces may balance—and as,

moreover, the resistance of the copper wire of the former was a comparatively large fraction, at least one sixth or one seventh, of the whole resistance in circuit with it, while the copper resistance of the larger primary coil was not much more than one fifteen-hundredth of the whole resistance in circuit with it—the ratio of the resistances was no doubt somewhat disturbed by the unequal heating of the two primary circuits, and was in reality rather less than what was inferred from the marked values of the coils used. That this was the case was shown by the fact that the apparent ratio decreased progressively from 40·3 to 39·45, as the strength of the testing current was diminished from its first value to rather less than one sixth. A better arrangement of the apparatus would have been to put the two primary wires in series with the battery, and to have connected the two secondaries in parallel circuit; but the matter was not thought important enough to require a repetition of the measurements.

The method of measuring coefficients of mutual induction described in this paper may perhaps be of use in the experimental study of dynamo-electric machines, whose whole action depends upon the variation of the coefficient of mutual induction between the field-magnet coils and the armature coils, as the latter take various positions during the course of a revolution.

XXIV. *An Account of Cauchy's Theory of Reflection and Refraction of Light.* By JAMES WALKER, M.A., Demonstrator at the Clarendon Laboratory, Oxford*.

THE theory of reflection and refraction of light holds such an important place among the problems of Optics which await their solution that it is advantageous to have a clear idea of the work which has been previously done in the subject.

The theory advanced by Green has been so thoroughly discussed by Lord Rayleigh and Sir W. Thomson that all questions connected with it may be considered as completely settled. But this is by no means the case with Cauchy's work on the subject; and some account of it may be of in-

* Read December 11, 1886.

terest, even though the theory cannot be said to contribute much towards a solution of the problem.

Several "reproductions"* of Cauchy's work have indeed appeared in French and German, but in most of them the elegance, and therewith the clearness, of Cauchy's method have been given up; while they leave in more or less obscurity the reasoning which led him to enunciate his "principle of continuity," and make no mention of a point of considerable interest, viz. the *mistake* which originally led to his adoption of a theory involving the strange assumption of a negative value for the coefficient of compressibility of the æther.

I.

Cauchy, at different periods, gave three distinct theories of reflection: the first two, however, require only a passing notice, as they were afterwards rejected by him as in no respect affording a complete solution of the problem.

The *first* theory was published in the *Bulletin de Férussac* of 1830. It rested on the true dynamical basis of the equality of pressures† at the interface of the media; but was vitiated by the neglect of the pressural waves, which must take part in the act of reflection and refraction. The method led, on the assumption of the equality of the density of the æther in the two media, to the formulæ given by Fresnel ‡.

The *second* theory was based on a method of obtaining the equations of condition at the interface, which was given in a lithographed memoir published in 1836. This method assumes a change in the equations of motion near the interface to a distance comparable with the radius of the sphere of activity of a molecule, and leads to the following theorem:—

"Etant donnés deux milieux ou deux systèmes de molécules séparés l'un de l'autre par le plan de yz , supposons que des équations d'équilibre ou de mouvement généralisées de manière à subsister pour tous les points de l'un et de l'autre système

* A. v. Ettingshausen, Pogg. *Ann.* l. p. 409; *Sitzb. der Wien. Akad.* xviii. p. 369. Beer, Pogg. *Ann.* xci. pp. 268, 467, 561; xcii. p. 402. Eisenlohr, Pogg. *Ann.* civ. p. 346. Briot, *Liouv. Journ.* (2nd) xi. p. 305; xii. p. 185. Lundquist, Pogg. *Ann.* clii. pp. 177, 398, 565.

† Cauchy's reasons for rejecting the principle of the equality of pressures at the interface are given in *Comptes Rendus*, xxviii. p. 60.

‡ Cauchy, *Mémoire sur la Dispersion*, § 10.

et même pour les points situés sur la surface de séparation, l'on puisse déduire une equation de la forme

$$\frac{d^2g}{dx^2} = \Theta,$$

g , Θ désignant deux quantités finies, mais variables avec les coordonnées xyz . On aura, pour $x=0$,

$$\frac{dg}{dx} = \frac{dg'}{dx}, \quad g = g',$$

en admettant que l'on prenne pour premier et pour second membre de chacune des formules les résultats que fournit la réduction de x à zéro, dans les deux valeurs de la fonction $\frac{dg}{dx}$ ou g relatives aux points intérieurs du premier et du second système."

The equations of condition resulting from the application of this theorem were published in Cauchy's memoir on Dispersion in the same year*. They express that the linear dilatation of the æther normal to the interface is the same for both the media, and that the rotations in the three coordinate planes of a particle at the interface is the same, whether the particle is considered as belonging to the first or second medium.

The method of deducing these conditions was given in a memoir presented to the French Academy on October 29, 1838†. This memoir has never been published; and all we know is that the method involved the assumption that the velocity of propagation of the pressural waves is very great compared with that of the distortional waves‡. In 1842 Cauchy showed that these conditions lead to Fresnel's formulæ§.

The *final* theory was published in detail|| in the years 1838 and 1839, and is contained in the 8th and 9th volumes of the *Comptes Rendus*, and in the *Exercices d'Analyse et de Physique*. Later volumes of the *Comptes Rendus* contain re-statements of it; and in 1850 an extension of the method

* *Mém. sur la Dispersion*, § 10.

† *Comptes Rendus*, vii. p. 751.

‡ *Ibid.* x. p. 905.

§ *Ibid.* xv. p. 418.

|| The idea seems to be prevalent that we are indebted to the German reproductions for our knowledge of the details of Cauchy's method.

was made to rotatory isotropic media* and to anisotropic media†; but this later work was never completed.

II.

Cauchy's final method‡ of determining the conditions at the interface of the media depended on finding the relations which must exist between the known values of the displacements in the interior of the medium, and the values, consistent with the conditions of the problem, which these displacements take when the change in the form of the equations of motion near the interface is taken into account.

Treating the æther as an isotropic elastic solid, for which the density is ρ , and the coefficients of compressibility and rigidity are k , n , the equations of motion are

$$\left. \begin{aligned} \rho \frac{d^2\xi}{dt^2} &= m \frac{d\delta}{dx} + n \nabla^2 \xi, \\ \rho \frac{d^2\eta}{dt^2} &= m \frac{d\delta}{dy} + n \nabla^2 \eta, \\ \rho \frac{d^2\zeta}{dt^2} &= m \frac{d\delta}{dz} + n \nabla^2 \zeta, \end{aligned} \right\} \quad . \quad . \quad . \quad (1)$$

where

$$\delta = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}, \quad \text{and} \quad m = k + \frac{1}{3}n.$$

Sir W. Thomson§ has shown that all possible solutions of these equations are included in

$$\xi = \frac{d\phi}{dx} + u, \quad \eta = \frac{d\phi}{dy} + v, \quad \zeta = \frac{d\phi}{dz} + w,$$

where ϕ, u, v, w are some functions of x, y, z, t and u, v, w such that $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$; further that, making these substitutions, equations (1) may be replaced by

$$\begin{aligned} \rho \frac{d^2\phi}{dt^2} &= (m+n) \nabla^2 \phi, & \rho \frac{d^2u}{dt^2} &= n \nabla^2 u, & \rho \frac{d^2v}{dt^2} &= n \nabla^2 v, \\ \rho \frac{d^2w}{dt^2} &= n \nabla^2 w. \end{aligned}$$

So that there are two modes of waves possible: a condensa-

* C. R. xxxi. pp. 160, 225.

† Ibid. xxxi. pp. 257, 297.

‡ Ibid. viii. pp. 374, 432, 459.

§ Baltimore Lectures, p. 32.

tional wave, propagated with velocity $\sqrt{\frac{m+n}{\rho}}$, and for which the velocity-potential is ϕ ; and a distortional wave, propagated with the velocity $\sqrt{\frac{n}{\rho}}$, and for which the components of the displacement are u, v, w .

Let the interface of the media be the plane of yz , and suppose the first medium on the side of positive x .

Considering only plane waves, which have the same period of vibration $2\pi/\omega$ and the same trace $by + cz = 0$ on the interface of the media, the values of ϕ, u, v, w satisfying the equations may be taken as

$$\phi = \frac{B_{II}}{\sqrt{-1}} e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} + \frac{C_{II}}{\sqrt{-1}} e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}}^*,$$

$$u = A e^{(ax + by + cz - \omega t)\sqrt{-1}} + A_I e^{(-ax + by + cz - \omega t)\sqrt{-1}},$$

$$v = B e^{(ax + by + cz - \omega t)\sqrt{-1}} + B_I e^{(-ax + by + cz - \omega t)\sqrt{-1}},$$

$$w = C e^{(ax + by + cz - \omega t)\sqrt{-1}} + C_I e^{(-ax + by + cz - \omega t)\sqrt{-1}};$$

where

$$a_{II} = +\sqrt{\frac{\rho\omega^2}{m+n} - b^2 - c^2}, \quad a = +\sqrt{\frac{\rho\omega^2}{n} - b^2 - c^2},$$

and $A, B, C; A_I, B_I, C_I$ are connected by the relations

$$Aa + Bb + Cc = 0, \quad -A_I a + B_I b + C_I c = 0.$$

The corresponding values of ξ, η, ζ are accordingly

$$\left. \begin{aligned} \xi &= A e^{(ax + by + cz - \omega t)\sqrt{-1}} + A_I e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II} a_{II} e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} \\ &\quad - C_{II} a_{II} e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}}, \\ \eta &= B e^{(ax + by + cz - \omega t)\sqrt{-1}} + B_I e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II} b e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} \\ &\quad + C_{II} b e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}}, \\ \zeta &= C e^{(ax + by + cz - \omega t)\sqrt{-1}} + C_I e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II} c e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}} \\ &\quad + C_{II} c e^{(-a_{II}x + by + cz - \omega t)\sqrt{-1}}. \end{aligned} \right\} \quad (2)$$

Now within the medium the displacements are those due to the distortional waves alone, and hence the values of ξ, η, ζ are

* The $\sqrt{-1}$ is inserted for convenience.

$$\left. \begin{aligned} \xi &= A e^{(ax+by+cz-\omega t)\sqrt{-1}} + A_1 e^{(-ax+by+cz-\omega t)\sqrt{-1}}, \\ \eta &= B e^{(ax+by+cz-\omega t)\sqrt{-1}} + B_1 e^{(-ax+by+cz-\omega t)\sqrt{-1}}, \\ \zeta &= C e^{(ax+by+cz-\omega t)\sqrt{-1}} + C_1 e^{(-ax+by+cz-\omega t)\sqrt{-1}}; \end{aligned} \right\} \quad (3)$$

whence, comparing these values with equations (2),

$$B_{11} = C_{11} = 0^*.$$

Near the interface, and for values of x less than the small quantity ϵ , the differential equations change form by the addition of terms whose coefficients are functions of x , which vanish when x exceeds the small quantity ϵ . These additional terms may be reduced to linear functions of ξ , η , ζ , and their differential coefficients with respect to x^\dagger , since the equations will still be satisfied by taking the displacements proportional to the same exponential $e^{(by+cz-\omega t)\sqrt{-1}}$.

We require now to determine the values of ξ , η , ζ which satisfy these altered equations.

Cauchy's method of doing this depends, as v. Ettingshausen† has pointed out, on the method of the variation of parameters: by this method the constants A , A_1 , ... are treated as functions of x ; and a first condition imposed upon them is that $\frac{d\xi}{dx}$, $\frac{d\eta}{dx}$, $\frac{d\zeta}{dx}$ must remain unaltered in form, so that

$$\left. \begin{aligned} \frac{d\xi}{dx} &= \left\{ A_1 a e^{(ax+by+cz-\omega t)\sqrt{-1}} - A_1 a e^{(-ax+by+cz-\omega t)\sqrt{-1}} \right. \\ &\quad \left. + B_{11} a_{11}^2 e^{(a_{11}x+by+cz-\omega t)\sqrt{-1}} + C_{11} a_{11}^2 e^{(-a_{11}x+by+cz-\omega t)\sqrt{-1}} \right\} \sqrt{-1}, \\ \frac{d\eta}{dx} &= \left\{ B_1 a e^{(ax+by+cz-\omega t)\sqrt{-1}} - B_1 a e^{(-ax+by+cz-\omega t)\sqrt{-1}} \right. \\ &\quad \left. + B_{11} a_{11} b e^{(a_{11}x+by+cz-\omega t)\sqrt{-1}} - C_{11} a_{11} b e^{(-a_{11}x+by+cz-\omega t)\sqrt{-1}} \right\} \sqrt{-1}, \\ \frac{d\zeta}{dx} &= \left\{ C_1 a e^{(ax+by+cz-\omega t)\sqrt{-1}} - C_1 a e^{(-ax+by+cz-\omega t)\sqrt{-1}} \right. \\ &\quad \left. + B_{11} a_{11} c e^{(a_{11}x+by+cz-\omega t)\sqrt{-1}} - C_{11} a_{11} c e^{(-a_{11}x+by+cz-\omega t)\sqrt{-1}} \right\} \sqrt{-1}. \end{aligned} \right\} \quad (4)$$

Consider now any one of the parameters, say B_{11} ; its value deduced from equations (2) and (4) is of the form

$$B_{11} = \left(\lambda \xi + \mu \eta + \nu \zeta + \pi \frac{d\xi}{dx} + \rho \frac{d\eta}{dx} + \sigma \frac{d\zeta}{dx} \right) e^{-(a_{11}x+by+cz-\omega t)\sqrt{-1}}.$$

Differentiating this equation with respect to x , and substitu-

* C. R. viii. p. 440.

† *Tom. cit.* p. 461.

† Pogg. *Ann.* l. p. 409.

ting for $\frac{d^2\xi}{dx^2}$, $\frac{d^2\eta}{dx^2}$, $\frac{d^2\zeta}{dx^2}$ from the changed differential equations, all the terms will cancel out except those which depend on the change of form; and we shall have

$$\frac{dB_{II}}{dx} = \left(L\xi + M\eta + N\zeta + P\frac{d\xi}{dx} + R\frac{d\eta}{dx} + S\frac{d\zeta}{dx} \right) e^{-(a_{II}x + by + cz - \omega t)\sqrt{-1}}$$

where L, M, \dots vanish for finite values of x .

Now the values of ξ, η, ζ will differ but slightly from those given by (3), so that this last expression may be written

$$\frac{dB_{II}}{dx} = (AL + BM + \dots)e^{(a - a_{II})x\sqrt{-1}} + (A_1L + B_1M + \dots)e^{-(a + a_{II})x\sqrt{-1}},$$

whence the variable part of B_{II} is

$$B_{II} = \int (AL + BM + \dots)e^{(a - a_{II})x\sqrt{-1}} dx \\ + \int (A_1L + B_1M + \dots)e^{-(a + a_{II})x\sqrt{-1}} dx.$$

Similar values are obtained for the parts of A, A_1, \dots which depend on x . Now L, M, \dots vanish for finite values of x ;

so that if $\int_0^x L dx, \int_0^x M dx, \dots$ are very small relatively to

$\lambda, \mu, \nu, \dots^*$, the variable part of B_{II} may be neglected if $-(a + a_{II})\sqrt{-1}$, $(a - a_{II})\sqrt{-1}$ have no real positive part; so that those among the coefficients A, A_1, \dots will remain unaltered, when the change in the medium near the interface is taken into account, which have the coefficient of x in their exponential factor with a real part *not less than* that of $a\sqrt{-1}$.

In the present case this will be so for all the parameters except B_{II} ; and hence, calling $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ the corrected values of ξ, η, ζ , we have

$$\bar{\xi} = Ae^{(ax + by + cz - \omega t)\sqrt{-1}} + A_1e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II}a_{II}e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}}, \\ \bar{\eta} = Be^{(ax + by + cz - \omega t)\sqrt{-1}} + B_1e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II}b_{II}e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}}, \\ \bar{\zeta} = Ce^{(ax + by + cz - \omega t)\sqrt{-1}} + C_1e^{(-ax + by + cz - \omega t)\sqrt{-1}} + B_{II}c_{II}e^{(a_{II}x + by + cz - \omega t)\sqrt{-1}},$$

* This necessitates, first, that the coefficients of the added terms in the altered differential equations are all finite, and their product by ϵ very small; secondly, that the thickness of the modified layer is small compared with the wave-length (*Comptes Rendus*, viii. p. 439; ix. p. 5).

$$\begin{aligned}\frac{d\bar{\xi}}{dx} &= \{Aae^{(ax+by+cz-\omega t)\sqrt{-1}} - A_1ae^{(-ax+by+cz-\omega t)\sqrt{-1}} \\ &\quad + B_{II}a_{II}^2e^{(a_{II}x+by+cz-\omega t)\sqrt{-1}}\}\sqrt{-1}, \\ \frac{d\bar{\eta}}{dx} &= \{Bae^{(ax+by+cz-\omega t)\sqrt{-1}} - B_1ae^{(-ax+by+cz-\omega t)\sqrt{-1}} \\ &\quad + B_{II}a_{II}be^{(a_{II}x+by+cz-\omega t)\sqrt{-1}}\}\sqrt{-1}, \\ \frac{d\bar{\xi}}{dx} &= \{Cae^{(ax+by+cz-\omega t)\sqrt{-1}} - C_1ae^{(-ax+by+cz-\omega t)\sqrt{-1}} \\ &\quad + B_{II}a_{II}ce^{(a_{II}x+by+cz-\omega t)\sqrt{-1}}\}\sqrt{-1};\end{aligned}$$

or, if

$\left. \begin{matrix} \xi, \eta, \xi \\ \xi_I, \eta_I, \xi_I \\ \xi_{II}, \eta_{II}, \xi_{II} \end{matrix} \right\}$ are the components of the displacements in the

$\left\{ \begin{matrix} \text{incident} \\ \text{reflected} \\ \text{pressural} \end{matrix} \right\}$ wave, $\bar{\xi}, \bar{\eta}, \bar{\xi}$ are such that

$$\bar{\xi} = \xi + \xi_I + \xi_{II}, \quad \bar{\eta} = \eta + \eta_I + \eta_{II}, \quad \bar{\xi} = \xi + \xi_I + \xi_{II};$$

$$\frac{d\bar{\xi}}{dx} = \frac{d\xi}{dx} + \frac{d\xi_I}{dx} + \frac{d\xi_{II}}{dx}, \quad \frac{d\bar{\eta}}{dx} = \frac{d\eta}{dx} + \frac{d\eta_I}{dx} + \frac{d\eta_{II}}{dx}, \quad \frac{d\bar{\xi}}{dx} = \frac{d\xi}{dx} + \frac{d\xi_I}{dx} + \frac{d\xi_{II}}{dx}.$$

In the same way, for the second medium the corrected values of the displacements are such that

$$\begin{aligned}\bar{\xi}' &= \xi' + \xi'', & \bar{\eta}' &= \eta' + \eta'', & \bar{\xi}' &= \xi' + \xi''; \\ \frac{d\bar{\xi}'}{dx} &= \frac{d\xi'}{dx} + \frac{d\xi''}{dx}, & \frac{d\bar{\eta}'}{dx} &= \frac{d\eta'}{dx} + \frac{d\eta''}{dx}, & \frac{d\bar{\xi}'}{dx} &= \frac{d\xi'}{dx} + \frac{d\xi''}{dx},\end{aligned}$$

where

$\left. \begin{matrix} \eta', \xi' \\ \eta'', \xi'' \end{matrix} \right\}$ are the components of the displacements in the $\left\{ \begin{matrix} \text{refracted} \\ \text{pressural} \end{matrix} \right\}$ wave.

Finally, assuming*, that for $x=0$,

$$\bar{\xi} = \bar{\xi}', \quad \bar{\eta} = \bar{\eta}', \quad \bar{\xi} = \bar{\xi}', \quad \frac{d\bar{\xi}}{dx} = \frac{d\bar{\xi}'}{dx}, \quad \frac{d\bar{\eta}}{dx} = \frac{d\bar{\eta}'}{dx}, \quad \frac{d\bar{\xi}}{dx} = \frac{d\bar{\xi}'}{dx}, \quad (5)$$

we have as the interfacial conditions, that for $x=0$,

$$\left. \begin{aligned}\xi + \xi_I + \xi_{II} &= \xi' + \xi'', & \eta + \eta_I + \eta_{II} &= \eta' + \eta'', \\ & \xi + \xi_I + \xi_{II} &= \xi' + \xi'', \\ \frac{d\xi}{dx} + \frac{d\xi_I}{dx} + \frac{d\xi_{II}}{dx} &= \frac{d\xi'}{dx} + \frac{d\xi''}{dx}, & \frac{d\eta}{dx} + \frac{d\eta_I}{dx} + \frac{d\eta_{II}}{dx} &= \frac{d\eta'}{dx} + \frac{d\eta''}{dx}, \\ & \frac{d\xi}{dx} + \frac{d\xi_I}{dx} + \frac{d\xi_{II}}{dx} &= \frac{d\xi'}{dx} + \frac{d\xi''}{dx}.\end{aligned}\right\} (6)$$

* C. R. ix, p. 94.

These equations express Cauchy's principle of the continuity of the motion of the æther, according to which the incident wave passes into the reflected and refracted waves "sans transition brusque."

Judging from the historical sequence of Cauchy's papers, there can be little doubt that he enunciated this principle as the physical interpretation of the result arrived at by reasoning analogous to the above; it is, however, impossible to agree with v. Ettingshausen that "Cauchy hat diese Gleichungen (6) anfänglich aus Gründen gerechtfertigt, die sich auf das Verfahren der Variationen der Constanten zurückführen lassen;"* as the principle is already involved in the assumption (5)†.

All that the above analysis really leads to, and all that Cauchy‡ claimed to have established by it, is the necessity for including the pressural waves in the problem of reflection and refraction.

Since the true dynamical equations of condition, given by the equality of displacements and pressures, are that for $x=0$,

$$\left. \begin{aligned} \bar{\xi} &= \bar{\xi}', & \bar{\eta} &= \bar{\eta}', & \bar{\zeta} &= \bar{\zeta}', \\ (m-n)\bar{\delta} + 2n\frac{d\bar{\xi}}{dx} &= (m'-n')\bar{\delta}' + 2n'\frac{d\bar{\xi}'}{dx}, \\ n\left(\frac{d\bar{\xi}}{dy} + \frac{d\bar{\eta}}{dx}\right) &= n'\left(\frac{d\bar{\xi}'}{dy} + \frac{d\bar{\eta}'}{dx}\right), & n\left(\frac{d\bar{\xi}}{dx} + \frac{d\bar{\zeta}}{dz}\right) &= n'\left(\frac{d\bar{\xi}'}{dx} + \frac{d\bar{\zeta}'}{dz}\right), \end{aligned} \right\} (7)$$

it is clear, as has been often pointed out, that Cauchy's assumption involves that of the identity of the statical properties of the æther in the two media. Lundquist§, however, considers that "Cauchy has established his principle of continuity by the aid of analysis, the exactitude of which it is

* *Sitzb. der Wien. Akad.* xviii. p. 371.

† I do not think Cauchy contemplated a continuous rapid transition of one medium into the other (*cf. C. R.* x. p. 347); neither does v. Ettingshausen in his paper. Supposing the assumption justified on these grounds, yet, as von der Mühl has pointed out, the former assumption respecting the coefficients of the additional terms in the modified equations precludes the assumption of a finite change in the statical properties of the media (*Matt. Ann.* v. p. 477).

‡ *C. R.* x. p. 347.

§ *Pogg. Ann.* clii. p. 185.

not easy to contest ;” and hence that this result, combined with the dynamically exact conditions (7), proves “the legitimacy of Green’s assumption of the equality of the compressibility and the rigidity of the æther in the two media.”

Cauchy himself did not see that this was involved in his conditions ; and so in what follows the compressibilities and rigidities of the two media will be considered as unequal.

III.*

Taking, as before, the interface of the media as the plane of yz , and the first medium on the side of positive x , let the axis of z be parallel to the plane of the waves, so that the plane of xy is the plane of incidence ; then, if $\xi \eta \zeta$ and $\xi' \eta' \zeta'$ denote the components of the displacements in the first and second medium respectively, $\xi \eta \zeta$, $\xi' \eta' \zeta'$ will be independent of z .

(1) Let the incident vibrations be perpendicular to the plane of incidence.

The general equations of motion are in this case

$$\rho \frac{d^2 \zeta}{dt^2} = n \left(\frac{d^2 \zeta}{dx^2} + \frac{d^2 \zeta}{dy^2} \right), \quad \rho' \frac{d^2 \zeta'}{dt^2} = n' \left(\frac{d^2 \zeta'}{dx^2} + \frac{d^2 \zeta'}{dy^2} \right),$$

and the principle of continuity gives for the interfacial conditions that for $x=0$,

$$\zeta = \zeta', \quad \frac{d\zeta}{dx} = \frac{d\zeta'}{dx}.$$

Assuming

$$\zeta = C e^{(ax+by-\omega t)\sqrt{-1}} + C' e^{(-ax+by-\omega t)\sqrt{-1}},$$

$$\zeta' = e^{(a'x+by-\omega t)\sqrt{-1}},$$

we get at once

$$C = \frac{a+a'}{2a}, \quad C' = \frac{a-a'}{2a},$$

$$\therefore C' = \frac{a-a'}{a+a'} = - \frac{\sin(i-r)}{\sin(i+r)},$$

since

$$\frac{b}{a} = \tan i, \quad \frac{b}{a'} = \tan r;$$

where i , r are the angles of incidence and refraction.

* C. R. viii. p. 985 ; ix. pp. 1, 59, 91, 676, 726, 727 ; x. p. 347. *Ex. d'An. et de Phys.* i. pp. 133, 212.

(2) Let the incident vibrations be in the plane of incidence. The equations of motion in the first medium are

$$\begin{cases} \rho \frac{d^2 \xi}{dt^2} = m \frac{d}{dx} \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} \right) + n \left(\frac{d^2 \xi}{dx^2} + \frac{d^2 \xi}{dy^2} \right), \\ \rho \frac{d^2 \eta}{dt^2} = m \frac{d}{dy} \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} \right) + n \left(\frac{d^2 \eta}{dx^2} + \frac{d^2 \eta}{dy^2} \right). \end{cases}$$

Using Green's* method of separating the distortional and condensational parts of the solution, and assuming

$$\xi = \frac{d\phi}{dx} + \frac{d\psi}{dy}, \quad \eta = \frac{d\phi}{dy} - \frac{d\psi}{dx},$$

the equations of motion become

$$\frac{d^2 \phi}{dt^2} = g^2 \left(\frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} \right), \quad \frac{d^2 \psi}{dt^2} = \gamma^2 \left(\frac{d^2 \psi}{dx^2} + \frac{d^2 \psi}{dy^2} \right),$$

where

$$g^2 = (m+n)/\rho, \quad \gamma^2 = n/\rho.$$

Similar equations apply to the second medium.

The principle of continuity gives for the interfacial conditions that for $x=0$,

$$\left. \begin{aligned} \frac{d\phi}{dx} + \frac{d\psi}{dy} &= \frac{d\phi'}{dx} + \frac{d\psi'}{dy} \\ \frac{d\phi}{dy} - \frac{d\psi}{dx} &= \frac{d\phi'}{dy} - \frac{d\psi'}{dx} \end{aligned} \right\} \dots \dots (8)$$

$$\left. \begin{aligned} \frac{d^2 \phi}{dx^2} + \frac{d^2 \psi}{dx dy} &= \frac{d^2 \phi'}{dx^2} + \frac{d^2 \psi'}{dx dy} \\ \frac{d^2 \phi}{dx dy} - \frac{d^2 \psi}{dx^2} &= \frac{d^2 \phi'}{dx dy} - \frac{d^2 \psi'}{dx^2} \end{aligned} \right\} \dots \dots (9)$$

Since these equations are true for all values of y , we may differentiate with respect to it, and hence, by means of the equations of motion, replace (9) by

$$\frac{1}{g^2} \frac{d^2 \phi}{dt^2} = \frac{1}{g'^2} \frac{d^2 \phi'}{dt^2}, \quad \frac{1}{\gamma^2} \frac{d^2 \psi}{dt^2} = \frac{1}{\gamma'^2} \frac{d^2 \psi'}{dt^2} \dots \dots (9a)$$

It may here be noted, that if we take the general equations of condition (7) and assume the equality of the rigidities of the æther in the two media with no assumption respecting the

* Collected Works, p. 261.

compressibilities, we get, instead of (9a),

$$\rho \frac{d^2\phi}{dt^2} = \rho' \frac{d^2\phi'}{dt^2}, \quad \rho \frac{d^2\psi}{dt^2} = \rho' \frac{d^2\psi'}{dt^2} \dots \dots \dots (9b)$$

Assume

$$\begin{aligned}\phi &= B_{//} e^{(a_{//}x + by - \omega t)\sqrt{-1}}, \\ \psi &= A e^{(ax + by - \omega t)\sqrt{-1}} + A' e^{(-ax + by - \omega t)\sqrt{-1}}, \\ \phi' &= B'' e^{(a''x + by - \omega t)\sqrt{-1}}, \\ \psi' &= e^{(a'x + by - \omega t)\sqrt{-1}}.\end{aligned}$$

The equations of motion give

$$\omega^2 = \gamma^2(a^2 + b^2) = g^2(a_{//}^2 + b^2) = \gamma'^2(a'^2 + b^2) = g'^2(a'^2 + b^2);$$

whence

$$a_{//} = b \sqrt{1 - \frac{\omega^2}{g^2 b^2}} \cdot \sqrt{-1} = b u_{//} \sqrt{-1}, \text{ say;}$$

$$a' = -b \sqrt{1 - \frac{\omega^2}{g'^2 b^2}} \cdot \sqrt{-1} = -b u' \sqrt{-1}, \text{ say,}$$

the negative sign being taken, as the second medium corresponds to negative x .

From equations (8) and (9a) we get

$$\left. \begin{aligned}u_{//} B_{//} \sqrt{-1} + (A + A') &= -u'' B'' \sqrt{-1} + 1, \\ b B_{//} - a(A - A') &= b B'' - a', \\ \frac{1}{g^2} B_{//} &= \frac{1}{g'^2} B'', \\ \frac{1}{\gamma^2} (A + A') &= \frac{1}{\gamma'^2} B''.\end{aligned} \right\} \quad (10)$$

The last two of these equations give

$$A + A' = \frac{\gamma^2}{\gamma'^2} B'' = \mu^2,$$

where μ is the refractive index, and

$$B_{//} = \frac{g^2}{g'^2} B'' = \frac{a'^2 + b^2}{a_{//}^2 + b^2} B'' = \mu_0^2 B'', \text{ say.}$$

Substituting in the first two of equations (10), we get

$$(u_{//} B_{//} + u'' B'') \sqrt{-1} = 1 - \mu^2,$$

whence

$$B'' = \frac{\mu^2 - 1}{\mu_0^2 u'' + u'''} \sqrt{-1},$$

and

$$\begin{aligned} A - A' &= \frac{a'}{a} + \frac{b}{a} (B'' - B') \\ &= \frac{a'}{a} + \frac{b}{a} (\mu^2 - 1) M \sqrt{-1}, \end{aligned}$$

where

$$M = \frac{\mu_0^2 - 1}{\mu_0^2 u'' + u'''}.$$

Hence

$$\begin{aligned} 2A &= \mu^2 + \frac{a'}{a} + \frac{b}{a} (\mu^2 - 1) M \sqrt{-1} \\ &= \frac{a'^2 + b^2}{a^2 + b^2} + \frac{a'}{a} + \frac{b}{a} \cdot \frac{a'^2 - a^2}{a^2 + b^2} \cdot M \sqrt{-1} \\ &= \frac{(aa' + b^2) + b(a' - a) M \sqrt{-1}}{a(a^2 + b^2)} (a' + a) = 2R e^{\delta \sqrt{-1}}, \text{ say;} \\ 2A' &= \frac{(aa' - b^2) - b(a' + a) M \sqrt{-1}}{a(a^2 + b^2)} \cdot (a' - a) = 2R' e^{\delta' \sqrt{-1}}, \text{ say.} \end{aligned}$$

Then R , R' denote the amplitudes of the incident and reflected vibrations, and δ , δ' the difference of phase between the incident and refracted, the reflected and refracted waves respectively.

Hence, if α is the azimuth with respect to the plane of incidence of the incident vibration, the reflected vibration will in general be elliptical with a difference of phase $\delta' - \delta$ between the components in and perpendicular to the plane of incidence; and if this difference of phase is destroyed, the azimuth β of the resulting rectilinear vibration will be given by

$$\cot \beta = R/C_r.$$

Hence

$$\frac{\cot \beta}{\cot \alpha} e^{(\delta' - \delta) \sqrt{-1}} = - \frac{(aa' - b^2) - b(a' + a) M \sqrt{-1}}{(aa' + b^2) + b(a' - a) M \sqrt{-1}}; \quad (11)$$

and

$$\begin{aligned} \frac{\cot^2 \beta}{\cot^2 \alpha} &= \frac{(aa' - b^2)^2 + b^2(a' + a)^2 M^2}{(aa' + b^2)^2 + b^2(a' - a)^2 M^2} \\ &= \frac{\cos^2(i + r) + M^2 \sin^2(i + r)}{\cos^2(i - r) + M^2 \sin^2(i - r)}; \end{aligned}$$

also

$$\tan(\delta_i - \delta) = \frac{M\{\tan(i+r) + \tan(i-r)\}}{1 - M^2 \tan(i+r) \tan(i-r)}.$$

Total Reflection.*

If μ is less than unity, we may write $\mu = \sin I$, and we get

$$a'^2 = \frac{4\pi^2}{\lambda'^2} \cos^2 r = \frac{4\pi^2}{\lambda^2} \sin(I-i) \sin(I+i).$$

Hence, if $i > I$, the value of a' becomes imaginary, and the refracted ray will die out as it leaves the refracting surface.

Writing

$$U = \sin^{\frac{1}{2}}(i-I) \sin(i+I),$$

we must substitute in the formulæ obtained above

$$a' = -\frac{2\pi}{\lambda} U \cdot \sqrt{-1},$$

the negative sign being taken, as the second medium is on the side of negative x .

Substituting this value, we find that the reflection is total both for the vibration in the plane of incidence, and for the vibration perpendicular to the plane of incidence, and for the difference of phase between the components of the reflected ray we get from (11)

$$\frac{\cot \beta}{\cot \alpha} \cdot e^{(\delta_i - \delta)\sqrt{-1}} = \frac{\sin i (\sin i + UM) + \cos i (M \sin i + U) \sqrt{-1}}{\sin i (\sin i + UM) - \cos i (M \sin i + U) \sqrt{-1}};$$

whence

$$\begin{aligned} \tan \frac{\delta_i - \delta}{2} &= \frac{\cos i \{M \sin i + \sin^{\frac{1}{2}}(i-I) \sin^{\frac{1}{2}}(i+I)\}}{\sin i \{\sin i + M \sin^{\frac{1}{2}}(i-I) \sin^{\frac{1}{2}}(i+I)\}} \\ &= \frac{M \frac{\sin^2 I}{\sin i} + \sin^{\frac{1}{2}}(i-I) \sin^{\frac{1}{2}}(i+I)}{\sin^2 i} \cdot \cos i, \end{aligned}$$

if the square and higher powers of the small quantity M are neglected.

Cauchy has $\sin^2 i$ instead of $\sin^2 I$ in the numerator of the

* *C. R.* ix. p. 764; xxx. p. 465.

last expression; the correct formula was first given by Beer*.

IV.†

Before proceeding further, it will be as well to discuss the value of the expression denoted above by M.

Cauchy, not seeing that his equations of condition involved the assumption of the identity of the statical properties of the æther in the two media, adopted the following relations,

$$m+n = -\epsilon^2 n, \quad m' + n' = -\epsilon'^2 n',$$

where ϵ, ϵ' are very small numerics.

These relations give

$$\mu_0^2 = \frac{m+n}{\rho} \times \frac{\rho'}{m'+n'} = \mu^2 \frac{\epsilon^2}{\epsilon'^2},$$

$$u_{||} = \sqrt{1 - \frac{n}{(m+n) \sin^2 i}} = \sqrt{1 + \frac{1}{\epsilon^2 \sin^2 i}} \doteq \frac{1}{\epsilon \sin i},$$

$$u'' = \doteq \frac{1}{\epsilon' \sin r}.$$

Hence

$$M = \frac{\mu^2 \frac{\epsilon^2}{\epsilon'^2} - 1}{\mu^2 \frac{\epsilon^2}{\epsilon'^2} \cdot \frac{1}{\epsilon \sin i} + \frac{1}{\epsilon' \sin r}} = \epsilon \sin i - \epsilon' \sin r = E \sin i,$$

if $E = \epsilon - \frac{\epsilon'}{\mu}$ ‡.

No attempt has been made, so far as I am aware, to indicate the reasons which led to Cauchy's adoption of the above remarkable relations between the coefficients of compressibility and rigidity of the æther in a medium.

In order to find a relation between the coefficients, Cauchy considered the condition which must be fulfilled if the incident light is completely polarized by reflection.

This condition is that $M=0$, giving since

$$M = \frac{\mu_0^2 - 1}{\mu_0^2 u_{||} + u''} = \frac{u_{||} - u''}{1 - u_{||} u''},$$

where $u_{||}, u''$ are both positive, that

$$u_{||} = u'', \text{ or } \frac{\rho}{m+n} = \frac{\rho'}{m'+n'}.$$

* Pogg. *Ann.* xci. p. 274.

† *C. R.* ix. pp. 691, 727.

‡ *C. R.* xxviii. p. 64. Originally Cauchy took $\epsilon=0$.

In his first memoir on the subject, Cauchy*, forgetting to take into account the fact of the media being on different sides of the plane of yz ; wrote

$$a'' = bu'' \sqrt{-1},$$

where u'' is positive.

Hence he obtained

$$M = \frac{\mu_0^2 - 1}{\mu_0^2 u_{||} - u''} = \frac{u_{||} + u''}{1 + u_{||} u''},$$

where $u_{||}$, u'' are both positive, giving as the condition for complete polarization

$$u_{||} = u'' = \infty, \text{ or } m + n = 0 = m' + n'.$$

He then argued that incomplete polarization must be due to the fact that these expressions differ slightly from zero, and that their value must be negative, in order that the pressural waves should be insensible at a distance from the interface for all angles of incidence.

In a memoir published in 1840 and in the *Exercices d'Analyse et de Physique*†, this mistake was corrected, and the true condition $\rho/(m+n) = \rho'/(m'+n')$ was given; but, apparently led astray by his original mistake and by a desire‡ (afterwards given up, *Compt. Rend.* xxviii. p. 125) to make complete polarization depend on the properties of the refracting medium alone, and not on any relation between the two media, he still adopted the solution

$$m + n = 0 = m' + n';$$

though he mentioned§ also the true solution, viz. that the coefficient of compressibility of the æther is infinite, and the wave-lengths of the pressural waves in the two media are equal.

Assuming that the æther is incompressible, the polarization of the reflected ray will be elliptical when the wave-lengths of the pressural waves are unequal, and we get

$$M = \left(\frac{\lambda_{||}^2}{\lambda'^2} - 1 \right) / \left(\frac{\lambda_{||}^2}{\lambda'^2} + 1 \right),$$

where $\lambda_{||}$, λ' are the wave-lengths of the pressural waves in the two media. This is Eisenlohr's suggestion; but the

* C. R. ix. p. 94. *Ex. d'An. et de Phys.* i. p. 167.

† C. R. x. p. 357. *Ex. d'An. et de Phys.* i. p. 233.

‡ C. R. ix. p. 727.

§ *Ibid.* x. p. 358.

form in which he made it does not show that it involves the absolute incompressibility of the æther. If $\lambda_{II}/\lambda'' = \lambda/\lambda'$, we get Green's formula. Eisenlohr* says that this assumption is absolutely untenable: it is, however, as Green shows, a direct consequence of the assumption made by him, and involved in Cauchy's conditions, viz. the identity of the statical properties of the æther in the two media.

Further than that, if we assume only the equality of the rigidities, the equations of condition become (8), (9b); whence

$$\frac{B_{II}}{B''} = \frac{\rho'}{\rho} = \mu^2;$$

and if the æther is incompressible,

$$M = \frac{\mu^2 - 1}{\mu^2 + 1}.$$

Haughton's† suggestion that the coefficient of compressibility is very great, but not infinite, does not help matters; so that it would appear that the only way to escape the difficulty is by one of Lord Rayleigh's‡ suggestions:—

(1) That “although the transition between the two media is so sudden that the principal waves of transverse vibrations are affected nearly in the same way as if it were instantaneous, yet we may readily imagine that the case is different for the surface-waves, whose existence is almost confined to the layer of variable density.”

(2) That “the densities concerned in the propagation of the so-called longitudinal waves are unknown, and may possibly not be the same as those on which transverse vibrations depend.”

Eisenlohr§ gives another (it appears entirely empirical) value for M : it involves, as Cauchy's, a negative value for the coefficient compressibility of the æther, and leads to formulæ closely agreeing with experiment; as, however, they contain a third disposable constant, this close agreement is hardly to be wondered at.

V.

Cauchy's formulæ for metallic reflection were originally published on April 15, 1839||, and thus were obtained from his

* Pogg. *Ann.* civ. p. 358.

† Phil. Mag. [4] iv. p. 81.

‡ *Ibid.* xlii. pp. 96, 97.

§ Pogg. *Ann.* civ. p. 356.

|| *C. R.* viii. p. 553.

second set of equations of condition, in which the pressural waves were neglected. The formulæ were republished on January 17, 1848*, and apparently no attempt was made to obtain equations in which the influence of the pressural waves was included.

Cauchy considers the peculiarities of metallic reflection to be due to a complex value of the refractive index.

Writing

$$\mu = \theta e^{\epsilon \sqrt{-1}},$$

we get

$$a'^2 = \frac{4\pi^2}{\lambda^2} (\theta^2 e^{2\epsilon \sqrt{-1}} - \sin^2 i) = \frac{4\pi^2}{\lambda^2} \cdot U^2 e^{2u \sqrt{-1}}, \text{ say;}$$

whence

$$U^2 \sin 2u = \theta^2 \sin 2\epsilon, \quad \cot \overline{2u - \epsilon} = \cot \epsilon \cos \left(2 \tan^{-1} \frac{\sin i}{\theta} \right). \quad (12)$$

Substituting

$$a' = -\frac{2\pi}{\lambda} U e^{u \sqrt{-1}}, \quad a = \frac{2\pi}{\lambda} \cos i, \quad b = \frac{2\pi}{\lambda} \sin i,$$

in the values of C_1/C , A_1/A , and making $M=0$, we get at once Cauchy's well-known formulæ.

Making these same substitutions in (11), we get

$$\frac{\cot \beta}{\cot \alpha} e^{\Delta \sqrt{-1}} = \frac{\sin^2 i + \cos i U e^{u \sqrt{-1}}}{\sin^2 i - \cos i U e^{u \sqrt{-1}}};$$

whence

$$\tan \Delta = \frac{2U \sin u \cos i \sin^2 i}{\sin^4 i - U^2 \cos^2 i} = \sin u \tan \left(2 \tan^{-1} \frac{U \cos i}{\sin^2 i} \right),$$

$$\frac{\cot^2 \beta}{\cot^2 \alpha} = \frac{\sin^4 i + \cos^2 i U^2 - 2 \sin^2 i \cos i U \cos u}{\sin^4 i + \cos^2 i U^2 + 2 \sin^2 i \cos i U \cos u},$$

$$= \cot(\psi - 45^\circ),$$

where

$$\cot \psi = \cos u \sin \left(2 \tan^{-1} \frac{U \cos i}{\sin^2 i} \right),$$

or, if $\alpha = 45^\circ$,

$$\cot 2\beta = \cos u \left(\sin 2 \tan^{-1} \frac{U \cos i}{\sin^2 i} \right).$$

At the polarizing angle I , for which $\Delta = \pi/2$, we have

$$U = \tan I \sin I, \quad u = 2\beta,$$

where β is the azimuth of the reflected vibrations, when the incident vibrations are in an azimuth 45° with respect to the plane of incidence.

These values substituted in equations (12) give the values of the constants θ , ϵ , and then these same equations serve for the determination of u , U for any other angle of incidence.

While the above equations can at the best be only considered incomplete, objections have also been made to the complex value of the refractive index involved in them.

Lord Rayleigh's criticism* that the real part of μ^2 should be positive, while the results of experiment substituted in Cauchy's equations give a value of μ^2 with its real part negative, seems not so much an argument against Cauchy's idea, as an "argument against the attempt to account for the effects on a purely elastic solid theory" †.

The value of μ^2 resulting from Sir W. Thomson's theory of light is a real negative quantity; this value substituted in Green's equations gives the reflection total at all angles of incidence. For this result there is no experimental evidence at present, except in the case of silver. The same will result from Lord Rayleigh's extension of Green's theory, unless, as seems scarcely probable, the refractive index of the pressural waves is a complex quantity.

VI.

In August 1850‡ Cauchy published the outlines of the result of applying his method to the case of reflection at the surface of an isotropic medium which possesses rotatory power.

The displacements in the upper medium are taken as

$$\xi = A b e^{(ax+by-\omega t)\sqrt{-1}} + A_1 b e^{(-ax+by-\omega t)\sqrt{-1}} + B_{11} \alpha_{11} e^{(a_1x+by-\omega t)\sqrt{-1}},$$

$$\eta = -A a e^{(ax+by-\omega t)\sqrt{-1}} + A_1 a e^{(-ax+by-\omega t)\sqrt{-1}} + B_{11} b e^{(a_1x+by-\omega t)\sqrt{-1}},$$

$$\zeta = \frac{2\pi}{\lambda} C e^{(ax+by-\omega t)\sqrt{-1}} + \frac{2\pi}{\lambda} C_1 e^{(-ax+by-\omega t)\sqrt{-1}};$$

* Phil. Mag. [4] xliii. p. 325.

† Eisenchr, Wied. Ann. i. p. 204; Glazebrook, Brit. Assoc. Report, 1885, p. 197.

‡ C. R. xxxi. pp. 160, 225.

and those in the lower medium, since there will be two refracted waves circularly polarized in opposite directions,

$$\begin{aligned}\xi' &= A_1' b e^{(a_1'x + by - \omega t)\sqrt{-1}} + A_2' b e^{(a_2'x + by - \omega t)\sqrt{-1}} + B'' a'' e^{(a''x + by - \omega t)\sqrt{-1}}, \\ \eta' &= -A_1' a_1' e^{(a_1'x + by - \omega t)\sqrt{-1}} - A_2' a_2' e^{(a_2'x + by - \omega t)\sqrt{-1}} + B'' b e^{(a''x + by - \omega t)\sqrt{-1}}, \\ \zeta' &= -\sqrt{-1} \cdot A_1' \frac{2\pi}{\lambda_1'} e^{(a_1'x + by - \omega t)\sqrt{-1}} + \sqrt{-1} A_2' \frac{2\pi}{\lambda_2'} e^{(a_2'x + by - \omega t)\sqrt{-1}}.\end{aligned}$$

Substituting these values in the equations of condition resulting from the principle of continuity, we get

$$\left. \begin{aligned}b(A + A_r - A_1' - A_2') &= B'' a'' - B_{II} a'', \\ -(A - A_r)a + A_1' a_1' + A_2' a_2' &= b(B'' - B_{II}), \\ b\{(A - A_r)a - A_1' a_1' - A_2' a_2'\} &= (B'' a''^2 - B_{II} a_{II}^2), \\ -(A + A_r)a^2 + A_1' a_1'^2 + A_2' a_2'^2 &= b(B'' a'' - B_{II} a''), \\ C + C_r &= \left(-\frac{\lambda}{\lambda_1'} A_1' + \frac{\lambda}{\lambda_2'} A_2'\right) \sqrt{-1}, \\ (C - C_r)a &= \left(-\frac{\lambda}{\lambda_1'} a_1' A_1' + \frac{\lambda}{\lambda_2'} a_2' A_2'\right) \sqrt{-1}.\end{aligned} \right\} \quad (13)$$

The last two of these equations give

$$\left. \begin{aligned}2aC &= \left\{ (a + a_2') \frac{\lambda}{\lambda_2'} A_2' - (a + a_1') \frac{\lambda}{\lambda_1'} A_1' \right\} \sqrt{-1}, \\ 2aC_r &= \left\{ (a - a_2') \frac{\lambda}{\lambda_2'} A_2' - (a - a_1') \frac{\lambda}{\lambda_1'} A_1' \right\} \sqrt{-1}.\end{aligned} \right\} \quad (14)$$

From the first and fourth we get

$$A + A_r = \frac{\lambda^2}{\lambda_1'^2} A_1' + \frac{\lambda^2}{\lambda_2'^2} A_2', \quad . \quad . \quad . \quad (15)$$

and from the second and third

$$B_{II} = \frac{a''^2 + b^2}{a_{II}^2 + b^2} B'' = \mu_0^2 B'';$$

whence, writing as before,

$$\frac{\mu_0^2 - 1}{\mu_0^2 a_{II} - a''} b = -M \sqrt{-1},$$

where M is the coefficient of ellipticity, and eliminating B_{II} B'' between the first two of equations (13),

$$\begin{aligned}(a - Mb \sqrt{-1})A - (a + Mb \sqrt{-1})A_r &= (a_1' - Mb \sqrt{-1})A_1' \\ &\quad + (a_2' - Mb \sqrt{-1})A_2',\end{aligned}$$

and from (15)

$$2aA = U_1 A_1' + U_2 A_2', \quad 2aA_1 = V_1 A_1' + V_2 A_2', \quad (16)$$

where

$$U_1 = \{ (aa_1' + b^2) + Mb(a_1' - a) \sqrt{-1} \} \frac{a_1' + a}{a^2 + b^2} \\ = \frac{2\pi}{\lambda_1'} \{ \cos(i - r_1) + M \sin(i - r_1) \sqrt{-1} \} \frac{\sin(i + r_1)}{\sin r_1},$$

$$V_1 = \{ (aa_1' - b^2) - Mb(a_1' + a) \sqrt{-1} \} \frac{a_1' - a}{a^2 + b^2} \\ = \frac{2\pi}{\lambda_1'} \{ \cos(i + r_1) - M \sin(i + r_1) \sqrt{-1} \} \frac{\sin(i - r_1)}{\sin r_1},$$

and U_2, V_2 are similar expressions with (2) written instead of (1) .

First, consider the case in which the incident vibrations are perpendicular to the plane of incidence.

Then $A = 0$, and equation (15) and the first of equations (16) give

$$A_1 = \frac{\lambda^2}{\lambda_1'^2} A_1' + \frac{\lambda^2}{\lambda_2'^2} A_2', \quad U_1 A_1' + U_2 A_2' = 0,$$

whence from (14)

$$\frac{A_1}{\frac{\lambda^2}{\lambda_1'^2} U_2 - \frac{\lambda^2}{\lambda_2'^2} U_1} = \frac{2a \sqrt{-1} C}{\frac{\lambda}{\lambda_2'} (a + a_2') U_1 + \frac{\lambda}{\lambda_1'} (a + a_1') U_2} \\ = \frac{2a \sqrt{-1} C_1}{\frac{\lambda}{\lambda_2'} (a - a_2') U_1 + \frac{\lambda}{\lambda_1'} (a - a_1') U_2},$$

and writing for $U_1, U_2, a, a_1', a_2', b$ their values in terms of the angles of incidence and refraction,

$$A_1 = -\sqrt{-1} \frac{\sin 2i \sin \frac{r_1 - r_2}{2} \cdot [\cos 2R - M \sin 2R \sqrt{-1}]}{D \left[\sin^2(i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot C, \\ C_1 = - \frac{D \sin(i - R) \sin(i + R) + D' \sin^2 \frac{r_1 - r_2}{2}}{D \left[\sin^2(i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot C;$$

where

$$D = \cos(i-R) + M \sin(i-R) \sqrt{-1},$$

$$D' = \cos(i+R) - M \sin(i+R) \sqrt{-1},$$

$$R = \frac{r_1 + r_2}{2}, \text{ the mean angle of refraction.}$$

Omitting squares and products of the small quantities M , $\sin \frac{r_1 - r_2}{2}$, the formulæ become

$$A_1 = -\sqrt{-1} \frac{\sin 2i \sin \frac{r_1 - r_2}{2} \cdot \cos 2R}{[\cos(i-R) + M \sin(i-R) \sqrt{-1}] \sin^2(i+R)} C,$$

$$C_1 = -\frac{\sin(i-R)}{\sin(i+R)} C.$$

Hence the reflected ray will be in general elliptically polarized, except for an angle of incidence such that the angle of mean refraction is $\pi/4$, in which case the reflected ray will be plane-polarized with vibrations perpendicular to the plane of incidence. In all cases the component perpendicular to the plane of incidence is practically the same as if the medium had no rotating power, the other component being very small.

Next consider the case in which the incident vibrations are in the plane of incidence.

Then $C=0$, and from equation (14)

$$A_1' = \frac{\lambda_1'}{\lambda_2'} \cdot \frac{a + a_2'}{a + a_1'} \cdot A_2',$$

and hence

$$\begin{aligned} \frac{C_1}{(a_1' - a_2')} &= \frac{\frac{\lambda^2}{\lambda_1' \lambda_2'} a A_1 \sqrt{-1}}{\frac{\lambda}{\lambda_2'} (a + a_2') V_1 + \frac{\lambda}{\lambda_1'} (a + a_1') V_2} \\ &= \frac{\frac{\lambda^2}{\lambda_1' \lambda_2'} a A \sqrt{-1}}{\frac{\lambda}{\lambda_2'} (a + a_2') U_1 + \frac{\lambda}{\lambda_1'} (a + a_1') U_2}. \end{aligned}$$

Whence

$$C_1 = \sqrt{-1} \cdot \frac{\sin 2i \sin \frac{r_1 - r_2}{2}}{D \left[\sin^2 (i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot A,$$

$$A_1 = \frac{D' \sin (i + R) \sin (i - R) + D \sin^2 \frac{r_1 - r_2}{2}}{D \left[\sin^2 (i + R) - \sin^2 \frac{r_1 - r_2}{2} \right]} \cdot A;$$

or, to the same degree of approximation as in the former case,

$$C_1 = \sqrt{-1} \sin 2i \frac{\sin \frac{r_1 - r_2}{2}}{[\cos (i - R) + M \sin (i - R) \sqrt{-1}] \sin^2 (i + R)} \cdot A,$$

$$A_1 = \frac{\cos (i + R) - M \sin (i + R) \sqrt{-1}}{\cos (i - R) + M \sin (i - R) \sqrt{-1}} \cdot \frac{\sin (i - R)}{\sin (i + R)} \cdot A.$$

Hence the reflected ray will be in general elliptically polarized, the component of the vibration in the plane of incidence being practically the same as if the refracting medium had no rotating power, the component of the vibration perpendicular to the plane of incidence being extremely small. At the polarizing angle for which $R + i = \pi/2$, the reflected vibration is plane-polarized, and the vibrations will be at an azimuth with respect to the plane of incidence given by

$$\tan \beta = \tan 2i \cdot \frac{\sin \frac{r_1 - r_2}{2}}{M}.$$

VII.

In the same year (1850) Cauchy extended his method to the problem of crystalline reflection: the complete solution was given in a memoir presented to the French Academy on September 16, 1850*.

This memoir was never published, though it was announced† to appear in the 23rd volume of the *Mémoires de l'Académie*; and we have only slight indications of Cauchy's manner of dealing with the problem.

* C. R. xxxi. p. 422.

† *Tom. cit.* p. 509.

In accordance with the results of his theory of double refraction, Cauchy does not suppose the vibrations to be necessarily strictly transversal and longitudinal*. In order to eliminate the amplitudes of the latter vibrations, he assumes as an approximation the strict transversality of the former, and thus obtains† four equations between the quasi-transversal amplitudes, which contain three coefficients, whose values are known when coordinate axes are taken depending on the refracting surface and the plane of incidence.

A second memoir‡ is devoted to the determination of the value of these coefficients, when fixed directions in the crystal are taken as the axes. The value of this determination is lessened by the fact, that at the very commencement an approximation is made depending on the peculiar relation between the coefficients of elasticity, which we have considered above.

This is all that has been published, except some notes indicating a few of the results of his analysis; it is, however, *probable*§ that Cauchy first obtained a solution on the assumption of the strict transversality of the luminous vibrations, and then proceeded to apply corrections to the values thus obtained, and it is *possible*|| that he adopted in the solution MacCullagh's idea of uniradial directions.

There is no need to enter further into this part of Cauchy's work, as Briot¶ has employed both these methods in his excellent adaptation of Cauchy's theory to the problem of Crystalline Reflection.

* *C. R.* xxxi. pp. 258, 299.

† *Tom. cit.* p. 297.

|| *Tom. cit.* p. 532.

† *Tom. cit.* p. 257.

§ *Tom. cit.* p. 160.

¶ *Liouv. Journ.* [2] xii. p. 185

